FRACTAL CURVATURES AND MINKOWSKI CONTENT OF SELF-CONFORMAL SETS

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ABSTRACT. For self-similar fractals, the Minkowski content and fractal curvature have been introduced as a suitable limit of the geometric characteristics of its parallel sets, i.e., of uniformly thin coatings of the fractal. For some self-conformal sets, the surface and Minkowski contents are known to exist. Conformal iterated function systems are more flexible models than similarities. This work unifies and extends such results to general self-conformal sets in $\mathbb{R}^d$. We prove the rescaled volume, surface area, and curvature of parallel sets converge in a Cesaro average sense of the limit. Fractal Lipschitz-Killing curvature-direction measures localize these limits to pairs consisting of a base point in the fractal and any of its normal directions. There is an integral formula.

We assume only the popular Open Set Condition for the first-order geometry, and remove numerous geometric assumptions. For curvatures, we also assume regularity of the Euclidean distance function to the fractal if the ambient dimension exceeds three, and a uniform integrability condition to bound the curvature on "overlap sets". A limited converse shows the integrability condition is sharp. We discuss simpler, sufficient conditions.

The main tools are from ergodic theory. Of independent interest is a multiplicative ergodic theorem: The distortion, how much an iterate of the conformal iterated function system deviates from its linearization, converges.

1. INTRODUCTION

Let $F \subseteq \mathbb{R}^d$ be a self-conformal set, i.e., invariant under finitely many contractions ϕ_i whose differentials $\phi_i'(x)$ are multiples of orthogonal matrices,

$$F = \bigcup_{i\in I} \{\phi_i x : x \in F\}.$$

Any self-similar set F is a simple, "rigid" special case. We assume the well-known Open Set Condition to keep overlaps $\phi_i F \cap \phi_j F$ small.

Classical volume and curvature are not meaningfully applicable to the geometry of F. So we approximate F with thin "coatings", its parallel sets

$$F_r := \left\{ x : \min_{y\in F} \left| x - y \right| \le r \right\},\ r > 0.$$

The limits of the rescaled Lebesgue measure and Lipschitz-Killing curvature-direction measures of F_r

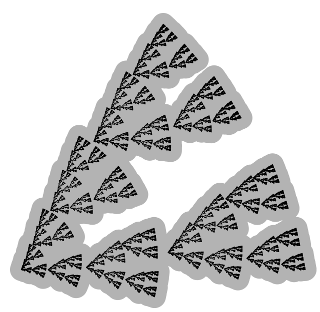

FIGURE 1.1.

Dark: self-conformal set F, IFS of three Möbius maps (6.0.1). Shaded: a parallel set F_r.

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do describe F geometrically. The purpose of this work is to make that statement precise under minimal assumptions.

The type of limit is most easily explained using the example of the well-known Minkowski content. We localize it as a measure version. Denote by $C_d(F_r, A)$ the Lebesgue measure of $F_r \cap A$ (if $A \subseteq \mathbb{R}^d$). The rescaling factor r^{D-d} keeps $r^{D-d} C_d(F_r, \cdot)$ from vanishing trivially as $r \to 0$. (Thermodynamic formalism determines for the correct dimension $D > 0$, see below.) But the rescaled total volume can oscillate. Cesaro averaging on a logarithmic scale suppresses this. Without any further assumptions, we prove the localized average Minkowski content converges (as a weak limit of measures):

$$\begin{aligned} C_d^{\mathrm{frac}}(F,\cdot) &:= \lim_{\epsilon\to 0} \frac{1}{|\ln \epsilon|} \int_\epsilon^1 r^{D-d} C_d(F_r,\cdot)\,\frac{dr}{r} \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-t(D-d)} C_d(F_{e^{-t}},\cdot)\,dt. \end{aligned}$$

Curvature and area fit into this approach as well as volume. Classically, they are intrinsically determined and form a complete system of certain Euclidean invariants. To illustrate classical curvature, let $K \subseteq \mathbb{R}^d$ be a compact body whose boundary ∂K is a smooth submanifold. Write $n(x)$ for the unit outer normal to K at $x \in \partial K$. The differential $n'(x)$ of the Gauss map n has real eigenvalues $\kappa_1(x), \ldots, \kappa_{d-1}(x)$ (geometrically, inverse radii of osculating spheres).[1] They are called principal curvatures of K at x. The full-dimensional Hausdorff measure $\mathscr{H}^{d-1}$ coincides with the Riemannian volume form on ∂K. Lipschitz-Killing curvature-direction measures are integrals of symmetric polynomials of principal curvatures: For integers $0 \le k < d$ and any Borel set $A \subseteq \mathbb{R}^d \times S^{d-1}$, define

$$C_k(K, A) := \mathrm{const} \int_{\partial K} 1\{(x, n(x)) \in A\} \sum_{j_1 < \cdots < j_{d-k-1}} \kappa_{j_1}(x) \cdots \kappa_{j_{d-k-1}}(x)\, d\mathscr{H}^{d-1}(x).$$

Our parallel sets F_r will only satisfy a positive reach condition rather than smoothness. (Parallel sets do not depend on the choice of ϕ_i and enjoy widespread use in singular curvature theory.) Their curvature-direction measures are defined via geometric measure theory. Each point x can have a different cone of normals. Anisotropic curvature quantities also prove useful to describe heterogeneous materials, see [STMK+11] and the references therein. Therefore, we prefer curvature-direction measures rather than non-directional Federer curvature, i.e., than their $\mathbb{R}^d$ marginal measures. We prove the average fractal curvature measures

$$C_k^{\mathrm{frac}}(F,\cdot) := \lim_{\epsilon\to 0} \frac{1}{|\ln \epsilon|} \int_\epsilon^1 r^{D-k} C_k(F_r,\cdot)\,\frac{dr}{r}$$

converge under mild assumptions.

While most literature uses this normalization, a different one displays more clearly $C_k^{\mathrm{frac}}(F,\cdot)$ does not depend on the ambient dimension (Remark 3.6 below).

[1] Curvature is second-order geometry because $n'(x)$ is a second-order coordinate derivative, as opposed to Minkowski content and various Hausdorff and packing measures.

We fix some notation for the assumptions. This dynamical system powers all convergence results: The left shift on F inverts the generating contractions, i.e., $\sigma\left(\phi_i x\right) := x$. Thermodynamic formalism defines both a unique dimension $D > 0$ and conformal probability measure μ on F. The contraction ratio is its potential, $\mu\left(\phi_{\omega_m}\circ\cdots\circ\phi_{\omega_1}F\right) = \int_F \left|\left(\phi_{\omega_m}\circ\cdots\circ\phi_{\omega_1}\right)'\right|^D d\mu$, $\omega_k \in I$. But the curvature is covariant under the similarity $\phi_i' x$ rather than σ. Also, only a similarity lets us compare its image of F_r locally with any thinner parallel set. (This restricts us to conformal ϕ_i.) The distortion ψ captures the mismatch between iterations of the dynamical system and their differentials (Figure 2.1). For an I-valued sequence ω and points $x, y \in F$, we prove convergence

$$\begin{aligned}\psi_{\widetilde{\omega|m},x}y &:= \left(\left(\phi_{\omega_m}\circ\cdots\circ\phi_{\omega_1}\right)'x\right)^{-1}\left(\phi_{\omega_m}\circ\cdots\circ\phi_{\omega_1}y - \phi_{\omega_m}\circ\cdots\circ\phi_{\omega_1}x\right),\\ \psi_{\tilde{\omega},x}y &:= \lim_{m\to\infty}\psi_{\widetilde{\omega|m},x}y.\end{aligned}$$

Two mild conditions guarantee $C_k^{\mathrm{frac}}(F,\cdot)$ converges (see Theorem 3.3). Neither condition applies to the volume ($k = d$), or to the surface area measure ($k = d-1$) if restricted to $\mathbb{R}^d$ (instead of $\mathbb{R}^d \times S^{d-1}$).

(1) Positive reach of the closed complement of almost every parallel set both of F and of almost every distorted fractal $\psi_{\omega,x}F$ (Assumption 3.1). This guarantees existence and continuity of curvature measures. It is automatically satisfied in ambient dimensions $d \le 3$. In higher dimensions, it is more convenient to check that almost all values of the Euclidean distance function are regular.

(2) A uniform integrability condition limits the C_k^{var} mass on overlaps of images under different ϕ_i (Assumption 3.2 below). This is sharp due to a limited converse (Remark 4.4). Let $B(x,r)$ be an r-ball around x. A sufficient, integrability condition is that

$$(r,\omega,x)\mapsto r^{-k}\sup_{M>0}\frac{1}{M}\sum_{m=0}^{M-1}C_k^{\mathrm{var}}\left(\left(\psi_{\widetilde{\omega|m},x}F\right)_r, \psi_{\widetilde{\omega|m},x}B(x,ar)\right),\ a>1$$

belongs to a Zygmund space related to μ. The much stronger, sufficient condition

$$\operatorname{ess\,sup}_{x\in F, r>0} r^{-k}C_k^{\mathrm{var}}(F_r, B(x,ar)) < \infty$$

intuitively means the rescaled principal curvatures should be uniformly bounded.

We are the first to prove average fractal curvature exists for general self-conformal sets because our C_k contains principal curvatures when $k < d-1$. Our theorem encompasses all previous convergence results about (D-dimensional) average Minkowski or surface content of self-conformal sets [KK10, FK11, Kom11]. We remove all previously needed geometrical conditions on the Open-Set-Condition set or on the hole structure. Fractal curvatures or their total mass were first investigated for self-similar sets [Win08, Zäh11, RZ10, WZ12, BZ11]. The Minkowski content was treated earlier [LP93, Fal95, Gat00], and in a very general context recently in [RW10] and [RW12].

There is an integral formula for the limit $C_k^{\mathrm{frac}}(F,\cdot)$ (Theorem 3.3). It is always finite. The formula shows the Minkowski content is strictly positive. But for $k < d$, the interpretation as fractal curvature should be checked if $C_k^{\mathrm{frac}}(F,\cdot)$ vanishes trivially [Win08].

Unsurprisingly, the fractal curvature is just as self-conformal as F. Its $\mathbb{R}^d$ marginal measure must therefore be a multiple of the conformal probability μ. That means the fractal locally looks the same almost everywhere. Its S^{d-1} sphere marginal measure inherits a group action invariance from the self-conformality (f_ω^{fib} (2.1.10) is invariant under $G_{\eta u}$ (2.1.9) below). Otherwise, it is free to provide geometric information beyond

its total mass. For example, the Sierpinski gasket resembles an equilateral triangle, and both have the same sphere marginals up to a constant [BZ11]. (But the group spreads it uniformly over S^{d-1} for generic ϕ_i, I. Future research could ask how curvature approaches its limit [LPW11].) In the self-similar case, $C_k^{\mathrm{frac}}(F,\cdot)$ is an (independent) product of both its marginal measures. In the self-conformal case, it is a skew product. But the skewing factor is easy to compute and only takes values in the orthogonal group of $\mathbb{R}^d$ (the factor $\left(\psi'_{\tilde{\eta},u}\hat{y}\right)^{\mathrm{orth}}$ in (2.1.10) below).

The motivation for this work is threefold. Fractal curvature is a geometric parameter to distinguish between sets of identical dimensions. Statistical estimates of various fractal dimensions already play a role in the applications. It is natural to expect further geometric parameters to be useful, both in their own right and to improve estimators for dimensions.

Secondly, curvature (geometry) determines analytic properties. In the classical, smooth Riemannian case, the Dirichlet Laplacian eigenvalue counting function has an expansion in terms of the boundary's curvatures. Heat kernel estimates also involve curvature. In one dimension, the next-to-leading order spectral asymptotics are proven: Let A be an open set whose boundary ∂A has a nontrivial, D-dimensional Minkowski content without Cesaro averaging (modified Weyl-Berry conjecture), as $\lambda \to \infty$:

$$\#\left\{\text{eigenvalues } \leq \lambda\right\} = \mathrm{const} C_1(A)\lambda^{1/2} + \mathrm{const}_D C_1^{\mathrm{frac}}\left(\partial A, A\times S^1\right)\lambda^{D/2} + o\left(\lambda^{D/2}\right).$$

Upper and lower bounds hold at the $\lambda^{D/2}$ order if the upper and lower Minkowski content are nontrivial [LP93]. In several dimensions, true curvature exists, and the conjecture fails. We hope to provide some missing geometric input.

Thirdly, there is the long-standing quest in geometric measure theory to extend the notion of curvature as far as possible. Classes of such "classical" sets include C^2-smooth manifolds, convex sets, sets of positive reach, and their locally finite unions [Fed59, Zäh86, RZ01]. See [Ber03a] and the references therein for an overview. Support measures generalize to all closed sets but loose additivity and other characteristic features of curvature [HLW04]. They and our fractal curvature usually live on mutually disjoint subsets of F, e.g. for the Sierpinski gasket.

The central ideas of the proof are explained in the respective section headings. The methods are dynamical although the results are geometric. We reduce the conformal case to similarities by separating ϕ into the distortion ψ and its derivative ϕ', which matches Euclidean covariance. The normal directions necessitate the two-sided shift dynamical system extended by the orthogonal group co-cycle $\left(\phi'\right)^{\mathrm{orth}}$.

2. Preliminaries

2.1. **Self-conformal sets and dynamics.** Conformal maps are automatically *(conjugate) holomorphic* in $\mathbb{R}^2$ or even *Möbius* in $\mathbb{R}^d$, $d>2$ (Liouville theorem, e.g. [Geh92, Theorem 1.5]).

Assumption 2.1. *Throughout this entire paper,* $\{\phi_i : i \in I\}$, $2 \leq |I| < \infty$ *will be a conformal iterated function system (IFS), i.e., each function*

$$\phi_i : V \to V$$

is an injective $C^{1+\gamma}$*-diffeomorphism defined on an open, connected set* $V \subseteq \mathbb{R}^d$ *such that the differentials* $\phi'(x)$ *are linear similarities of* $\mathbb{R}^d$*, and*

$$\left|\phi_i x - \phi_i y\right| \leq s_{\max}\left|x-y\right|, \; x,y \in V$$

for some global $s_{\max} < 1$. *We assume the Open Set Condition (OSC): There is an open, connected, bounded, nonempty set* $\mathring{X}$ *such that* $X \subseteq V$,

$$\phi_i \mathring{X} \subseteq \mathring{X},\ i \in I$$

and

$$\left(\phi_i \mathring{X}\right) \cap \left(\phi_j \mathring{X}\right) = \emptyset,\ i \neq j,\ i,j \in I.$$

Without loss of generality ([PRSS01]), the set X shall satisfy the *Strong Open Set Condition*

$$\mathring{X} \cap F \neq \emptyset.$$

The IFS shall extend conformaly in $C^{1+\gamma}$ to the closure of an open set $V_{MU} \supseteq \overline{V}$: $\phi_i : \overline{V_{MU}} \to \overline{V_{MU}}$.

This is the situation studied in [MU96], except we allow only finitely many maps ϕ_i. Their cone condition is not needed for finite I, and their bounded distortion condition can be proved [MU03]. See [MU96] for further discussion and facts given without a reference.

The *self-conformal fractal* is the unique, invariant, compact set $\emptyset \neq F = \bigcup_{i\in I} \phi_i F \subseteq X$. The (Borel probability) *conformal measure* μ on F and *dimension* $D > 0$ are uniquely characterized by

$$\int_F f(x)\, d\mu(x) = \int_F \sum_{i\in I} \left|\phi_i' x\right|^D f\left(\phi_i x\right)\, d\mu(x). \tag{2.1.1}$$

Abbreviate $x|n := x_1 \dots x_n \in I^n$, the *reversed word* $\widetilde{x|n} := x_n \dots x_1$, $\phi_{x|n} := \phi_{x_1} \circ \dots \circ \phi_{x_n}$, $I^* := \bigcup_{n\in\mathbb{N}_0} I^n$. We will identify μ-almost all points $x \in F$ with their (unique) coding sequence $x_1 x_2 \dots \in I^{-\mathbb{N}}$, i.e., $x \equiv \lim_{n\to\infty} \phi_{x|n}(v)$ for any starting $v \in V$. The *(left) shift* $\sigma : F \to F$ or $\sigma : I_F^{\mathbb{N}} \to I_F^{\mathbb{N}}$ maps x to $\phi_{x_1}^{-1} x$. The unique, equivalent, *σ-invariant probability* ν is ergodic. Denote its Hölder continuous density $p := d\nu/d\mu$. The proofs need the *two-sided shift space* $I^{\mathbb{N}} \times F$. Let $\sigma_{\mathrm{bi}}(\omega_1\omega_2\omega_3\dots, x_1x_2x_3\dots) := (x_1\omega_1\omega_2\dots, x_2x_3x_4\dots)$ for $(\omega, x) \in I^{\mathbb{N}} \times F$; perhaps it is most natural to think of ω in reverse order: $\sigma_{\mathrm{bi}}((\dots\omega_2\omega_1)(x_1x_2\dots)) = ((\dots\omega_1 x_1)(x_2x_3\dots))$. Denote μ_{bi}, ν_{bi}, $p_{\mathrm{bi}} = d\nu_{\mathrm{bi}}/d\mu_{\mathrm{bi}}$ the Rokhlin extensions.

We extend it again, by the orthogonal group, to $I^{\mathbb{N}} \times F \times O(d)$. The new measure $\underline{\nu_{\mathrm{bi}}} := \nu_{\mathrm{bi}} \otimes \mathscr{H}_{O(d)}$ is the direct product with the Haar probability on $O(d)$. The new left shift is

$$\underline{\sigma_{\mathrm{bi}}}^m(\omega, x, g) := \left(\widetilde{x|m}.\omega,\ \sigma^m x,\ \left(\left(\phi_{x|m}' \sigma^m x\right)^{\mathrm{orth}}\right)^{-1} g\right), \tag{2.1.2}$$

i.e., the *skew product* with the *Rokhlin co-cycle* $(\omega, x) \mapsto \left(\phi_{x|1}' \sigma x\right)^{\mathrm{orth}-1}$.

Recall some basic properties. The Perron-Frobenius operator gives ([PU10, Lemma 4.2.5]):

$$\int_F \frac{g}{p}(x)\, d\nu(x) = \int_{I^{\mathbb{N}}\times F} \frac{g}{p}\left(\phi_{\widetilde{\omega|n}} x\right) d\nu_{\mathrm{bi}}(\omega, x),\ g \in L_1(\mu). \tag{2.1.3}$$

Denote $B(x,r) := \left\{y : \left|y - x\right| \leq r\right\}$. F is a D-set for μ (*Ahlfors regular*): there is a $0 < C_F < \infty$,

$$C_F^{-1} \le \frac{\mu B(x,r)}{r^D} \le C_F. \tag{2.1.4}$$

Bounded distortion is fundamental. There is a global constant $0 < K < \infty$ such that for all $x, y \in \overline{V_{MU}}$, $\tau \in I^*$,

$$K^{-1} \le \frac{\left|\phi'_\tau x\right|}{\left|\phi'_\tau y\right|} \le K. \tag{2.1.5}$$

There is a constant $0 < K_\phi < \infty$ such that for all $x \in V_{MU}$, $r/\left|\phi'_\tau x\right| \le \operatorname{dist}\left(x, V_{MU}^c\right)$, $\tau \in I^*$, [MU96, (BDP.3)]:

$$B\left(\phi_\tau x, rK_\phi^{-1}\right) \subseteq \phi_\tau B\left(x, \frac{r}{\left|\phi'_\tau x\right|}\right) \subseteq B\left(\phi_\tau x, rK_\phi\right). \tag{2.1.6}$$

The *distortion*, how much the non-linear map $\phi_{\omega|n}$ deviates from its differential (up to isometries), converges exponentially (Proposition 4.8 below):

$$\psi_{\widetilde{\omega|n},x} y \quad := \quad \left(\phi'_{\widetilde{\omega|n}} x\right)^{-1} \left(\phi_{\widetilde{\omega|n}} y - \phi_{\widetilde{\omega|n}} x\right), \tag{2.1.7}$$

$$\psi_{\tilde{\omega},x} y \quad := \quad \lim_{n\to\infty} \psi_{\widetilde{\omega|n},x} y. \tag{2.1.8}$$

Note its derivative is point-wise a similarity. The *Brin ergodicity group* is the compact subgroup of $O(d)$ generated by

$$G_{\eta u} := \overline{\left\langle \left(\left(\psi'_{\tilde{\eta},u} x\right)^{\mathrm{orth}}\right)^{-1} \left(\phi'_i x\right)^{\mathrm{orth}} \left(\psi'_{\tilde{\eta},u} \phi_i x\right)^{\mathrm{orth}} : x \in F, i \in I \right\rangle_{O(d)}}. \tag{2.1.9}$$

Let $\mathscr{H}_G$ be its Haar probability. For $f : V \times S^{d-1} \to \mathbb{R}$, the average of f/p over the *ergodic fibre* through (ω, x) is

$$f_\omega^{\mathrm{fib}}(z,n) := \int_F \int_{G_{\eta u}} f\left(\hat{y}, \left(\psi'_{\tilde{\eta},u}\hat{y}\right)^{\mathrm{orth}-1} \hat{g} \left(\psi'_{\tilde{\omega},u} z\right)^{\mathrm{orth}} n\right) d\mathscr{H}_G\left(\hat{g}\right) d\mu\left(\hat{y}\right). \tag{2.1.10}$$

Different choices of $(\eta, u) \in I^{\mathbb{N}} \times F$ do not alter f_ω^{fib} (but do conjugate $G_{\eta u}$). So $G_{\eta u}$ and f_ω^{fib} can be skipped when replacing u with x. Readers not interested in directional C_k should ignore all underbars and set $G_{\eta u} := O(d)$, $f_\omega^{\mathrm{fib}}(z,n) := \int_F f\,d\mu$.

2.2. **Curvature.** We will approximate F with its r*-parallel set* (Minkowski sausage, dilation, offset, homogenization) F_r or the closure of the complement:

$$F_r := \left\{x \in \mathbb{R}^d : \left|x - y\right| \le r \text{ for a } y \in F\right\}, \qquad \widetilde{F}_r := \overline{(F_r)^c}. \tag{2.2.1}$$

The *reach* of a closed set $K \subseteq \mathbb{R}^d$ is the supremum of all $s > 0$ such that every point $x \in K_s$ has a unique nearest neighbor $y \in K$. If reach $K > 0$, the set of pairs $\left(y, \frac{x-y}{|x-y|}\right)$ forms the Federer *normal bundle*, nor K [Fed59]. Positive reach replaces the C^∞ smoothness often assumed in differential geometry.

We will implicitly extend any conformal or similarity map ξ of $\mathbb{R}^d$ to one of $\mathbb{R}^d \times S^{d-1}$, $\xi(z,n) := \left(\xi(z), \xi'(z)^{\mathrm{orth}} n\right)$ (i.e., to the cotangent bundle equipped with the Sasaki metric). Here, ξ'^{orth} is the *orthogonal group component* of its derivative ξ'.

We will need only these properties of *Lipschitz-Killing curvature-direction measures* C_k: Let $K \subseteq \mathbb{R}^d$ be a set of positive reach. For $k \in \{0, \ldots, d-1\}$, $C_k(K, \cdot)$ is a signed Borel measure on nor $K \subseteq \mathbb{R}^d \times S^{d-1}$ with (locally) finite variation. It is Euclidean motion

covariant and *homogeneous* of degree k, i.e., if g is a similarity with ratio $s>0$ and orthogonal group part g^{orth}, then for Borel $A\subseteq\mathbb{R}^d$, $N\subseteq S^{d-1}$,

$$C_k\left(gK,gA\times g^{\text{orth}}N\right)=s^kC_k(K,A\times N). \tag{2.2.2}$$

It is *locally determined*, i.e., if two sets of positive reach K, $\hat{K}$ agree $K\cap G=\hat{K}\cap G$ on an open set $G\subseteq\mathbb{R}^d$, then for $A\subseteq\mathbb{R}^d$, $N\subseteq S^{d-1}$,

$$C_k(K,(A\cap G)\times N)=C_k\left(\hat{K},(A\cap G)\times N\right). \tag{2.2.3}$$

A special case of *continuity* is stated in Fact 4.15 below. The $\mathbb{R}^d$ projection of $C_{d-1}(K,\cdot)$ agrees with half the $(d-1)$-dimensional Hausdorff measure $\mathscr{H}^{d-1}$ *(surface area)* on the boundary ∂K. Because the *Lebesgue* measure $\mathscr{L}^d$ and the rotation invariant probability $\mathscr{H}_{S^{d-1}}$ on the sphere share these properties, we define for Borel $A\subseteq\mathbb{R}^d$, $N\subseteq S^{d-1}$,

$$C_d(K,A\times N):=\mathscr{L}^d(K\cap A)\mathscr{H}_{S^{d-1}}(N).$$

(The trivial product with $\mathscr{H}_{S^{d-1}}$ makes every C_k live on $\mathbb{R}^d\times S^{d-1}$.) We define the curvature measure of F_r via the normal reflection since we assume $0<\operatorname{reach}\widetilde{F_r}$ (Assumption 3.1 below): for Borel $A\subseteq\mathbb{R}^d$, $N\subseteq S^{d-1}$, $k<d$,

$$C_k(F_r,A\times N):=(-1)^{d-1-k}C_k\left(\widetilde{F_r},A\times\{-n:n\in N\}\right). \tag{2.2.4}$$

This is consistent in case both F_r and $\widetilde{F_r}$ are (unions of) sets of positive reach ([RZ01, Theorems 3.3, 3.2]). Of course, $C_k(F_r,\cdot)$ enjoys the above geometrical properties. All assertions in this paper also hold for (non-directional) *Federer curvature measures*, i.e., their projection $C_k\left(F_r,\cdot\times S^{d-1}\right)$ onto $\mathbb{R}^d$. (We conjecture projecting to $\mathbb{R}^d$ does not reduce the variation's mass [BZ11, Rem. 3.15].)

For a gentle introduction to curvature, combine the brief summary in [Zäh11] with [SW92] (polyconvex case), [KP08] (currents and Hausdorff measure), [Fed59] (positive reach), [Zäh86] (curvature-direction measure), [Ber03a] (survey of notions).

To put curvature in context: The total masses $K\mapsto C_k\left(K,\mathbb{R}^d\times S^{d-1}\right)$ form a complete system of Euclidean invariants in the following sense. Every set-additive, continuous, motion invariant functional on the space of convex bodies is a linear combination of total curvatures (Hadwiger's theorem). By means of an approximation argument, this holds for large classes of singular sets, including sets of positive reach [Zäh90]. Furthermore, C_k is the integral of symmetric functions of generalized principal curvatures κ_i over the $d-1$-dimensional Hausdorff measure $\mathscr{H}^{d-1}$ on the normal bundle:

$$C_k(K,A\times N)=c_{k,d}\int_{(\operatorname{nor}K)\cap A\times N}\frac{\sum_{i_1<\dots<i_{d-1-k}}\kappa_{i_1}(x,n)\dots\kappa_{i_{d-1-k}}(x,n)}{\prod_{j=1}^{d-1}\sqrt{1+\kappa_j^2(x,n)}}\,d\mathscr{H}^{d-1}(x,n).$$

2.3. **Remaining notation.** Fix arbitrary constants $a>1$ and $0<\epsilon_{\max}\le a^{-1}\operatorname{dist}(F,V^c)$. Let $(\eta,u)\in I^{\mathbb{N}}\times F$ be any fixed reference point. Define for $x,z\in V$, $r>0$,

$$A(|x-z|,r):=\max\left(1-\frac{|x-z|}{ra},0\right) \tag{2.3.1}$$

a "smoothed-out" indicator function of the ar-ball around x. Inspired by $R(x,m)\sim\operatorname{dist}\left(x,\partial\bigcup_{\tau\in I^m}\phi_\tau X\right)$, the *renewal epochs* are defined (Definition 4.10) for μ-almost all

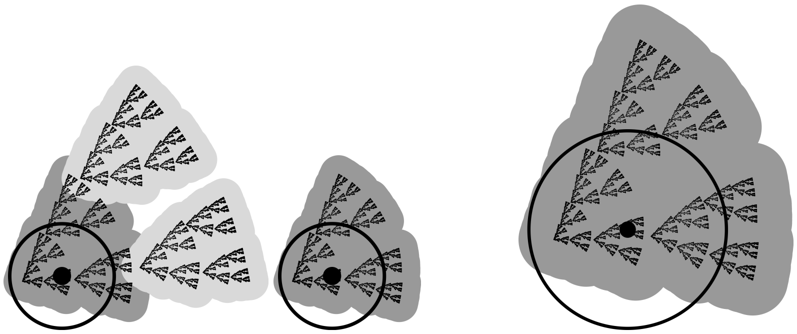

FIGURE 2.1. Left: fractal F (black), parallel set F_r (both shades), point $x=(1,1,2,3,1,\dots)$ (marked) surrounded by a ball of $2a$ times its renewal radius $R(x,1)$. The ball supports A and localizes curvature, so it must avoid $\left(\phi_2 F\right)_r \cup \left(\phi_2 F\right)_r$ (light gray).
Middle: level set $\left(\phi_1 F\right)_r$ containing x.
Right: Magnification is the composition of the dynamical system map σ and distortion by ψ. Black/shaded: magnified $\left(\phi_1 F\right)_r$ or $\left(\psi_{1,x} F\right)_{r/\left|\phi_1' x\right|}$. The marked point $\psi_{1,x}\sigma x$ is surrounded with $2aR(\sigma x,0)$.

$x\in F$ and $m\in\mathbb{N}$ as:

$$\begin{aligned}\widetilde{R}(x,0) &:= \frac{\operatorname{dist}(x,X^c)}{2K_\phi a}\min\left\{1,\frac{2K_\phi^{-1}a\epsilon_{\max}}{\max_{x\in F}\operatorname{dist}(x,X^c)}\right\},\\ R(x,m) &:= \sup_{n\ge m}\left|\phi_{x|n}'\sigma^n x\right|\widetilde{R}(\sigma^n x,0).\end{aligned}\tag{2.3.2}$$

Denote the measure theoretic *entropy* of ν and σ,

$$H_\nu := -D\int_F \ln\left|\phi_{x_1}'\sigma x\right| d\nu(x).\tag{2.3.3}$$

Define for $x\in F$, $r,s,t\in(0,\epsilon_{\max}]$, $f: V\times S^{d-1}\to\mathbb{R}$ Borel measurable, with the shorthand $\underline{z}=(z,n)\in\mathbb{R}^d\times S^{d-1}$, the measure

$$\rho_{(s,t]}^{k,\pm}(x,r,f) := \int_{\operatorname{nor}\widetilde{F_r}} \frac{1_{(s,t]}(r)\, r^{D-k} A(|x-z|,r)}{\int A\left(\left|y-z\right|,r\right) d\mu(y)} f(\underline{z})\, dC_k^{\pm}(F_r,\underline{z}),\tag{2.3.4}$$

and for a finite word $\tau\in I^*$ in (2.1.7) and an infinite word $\omega\in I^{\mathbb{N}}$ in (2.1.8):

(2.3.5) $\pi_\tau^{k,\pm}(x,r,f) :=$

$$\int \frac{1_{(R(x,1),R(x,0)]}\left(\frac{r}{\left|\psi_{\tau,u}'x\right|}\right) r^{D-k} A\left(\left|\psi_{\tau,u}x-\psi_{\tau,u}z\right|,r\right)}{\int\left|\psi_{\tau,u}'y\right|^D A\left(\left|\psi_{\tau,u}y-\psi_{\tau,u}z\right|,r\right) d\mu(y)} f(\underline{z})\, dC_k^{\pm}\left(\left(\psi_{\tau,u}F\right)_r,\psi_{\tau,u}\underline{z}\right),$$

(2.3.6) $\pi^k_{\tilde{\omega}}(x,r,f) :=$

$$\int \frac{1_{(R(x,1),R(x,0)]}\left(\frac{r}{\left|\psi'_{\tilde{\omega},u}x\right|}\right) r^{D-k} A\left(\left|\psi_{\tilde{\omega},u}x-\psi_{\tilde{\omega},u}z\right|,r\right)}{\int\left|\psi'_{\tilde{\omega},u}y\right|^{D} A\left(\left|\psi_{\tilde{\omega},u}y-\psi_{\tilde{\omega},u}z\right|,r\right)\,d\mu(y)} f(\underline{z})\,dC_k\left(\left(\psi_{\tilde{\omega},u}F\right)_r,\psi_{\tilde{\omega},u}\underline{z}\right)$$

if these integrals exist, and

$$\rho^k_{(s,t]} := \rho^{k,+}_{(s,t]} - \rho^{k,-}_{(s,t]}, \qquad \pi^k_\tau := \pi^{k,+}_\tau - \pi^{k,-}_\tau,$$
$$\rho^{k,\mathrm{var}}_{(s,t]} := \rho^{k,+}_{(s,t]} + \rho^{k,-}_{(s,t]}, \qquad \pi^{k,\mathrm{var}}_\tau := \pi^{k,+}_\tau + \pi^{k,-}_\tau.$$

The above transform into each other, see (4.5.4). The proofs define further symbols: K_ϕ (2.1.6), c_ψ K_ψ Proposition 4.8, C_A (4.4.3), q (4.5.3).

3. Main result and Limit formula

Recall $F=\bigcup_{i\in I}\phi_i F$ is self-conformal with (OSC), see Assumption 2.1.

Assumption 3.1. *(Regularity of parallel sets) Throughout this entire paper, whenever $k<d$, we will assume the following. Lebesgue almost all $r>0$ shall be regular values of the Euclidean distance function to F and to $\psi_{\tilde{\omega},u}F$ for μ_{bi}-almost every ω.*

Alternatively, let almost all $r>0$, $\omega\in I^{\mathbb{N}}$ satisfy

(1) reach $\widetilde{F}_r>0$,
(2) *if* $(y,m)\in\operatorname{nor}\widetilde{F}_r$, *then* $(y,-m)\notin\operatorname{nor}\widetilde{F}_r$,
(3) reach $\left(\psi_{\tilde{\omega},u}F\right)_r>0$.

If the surface area measure C_{d-1} is restricted to $\mathbb{R}^d$ instead of $\mathbb{R}^d\times S^{d-1}$, the assumption is needed only for $k<d-1$.

This is always satisfied in ambient dimensions $d\le 3$ due to [Fu85]. It is also true for any strictly self-similar set whose convex hull is a polytope ([Pok11]). Regular distance values imply the alternative conditions, see [Fu85, Theorem 4.1], [RZ03, Proposition 3].

Assumption 3.2. *(Uniform integrability) Assume at least one of the following:*

(1) $k=d$ *(Minkowski content),*
(2) $k=d-1$ *(Surface area content),*
(3)
$$\operatorname{ess\,sup}_{r>0,x\in F} r^{-k} C^{\mathrm{var}}_k\left(F_r,\mathring{B}(x,ar)\right)<\infty,$$
μ-essential in x, Lebesgue in r.
(4) *(Zygmund space)*
$$h(r,\omega,x) := r^{-k}\sup_{M>0}\frac{1}{M}\sum_{m=0}^{M-1} C^{\mathrm{var}}_k\left(\left(\psi_{\widetilde{\omega|m},u}F\right)_r,\psi_{\widetilde{\omega|m},u}\mathring{B}(x,ar)\right)$$
satisfies both
$$\int_{I^{\mathbb{N}}\times F}\int_{R(x,1)}^{R(x,0)} \max\{0,h(r,\omega,x)\ln h(r,\omega,x)\}\,\frac{dr}{r}\,d\nu_{\mathrm{bi}}(\omega,x)<\infty,$$
$$\int_{I^{\mathbb{N}}\times F}\int_{0}^{\epsilon_{\max}} \sup_{0<\epsilon\le\frac{\epsilon_{\max}}{2}} \frac{1_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(r)\,C^{\mathrm{var}}_k\left(F_r,\mathring{B}(x,ar)\right)}{\ln(\epsilon_{\max}/\epsilon)\,r^k}\,\frac{dr}{r}\,d\nu_{\mathrm{bi}}(\omega,x)<\infty.$$

(5) *As above, except integrating $h\ln h$ from $R(x,1)/K$ to $R(x,0)K$ and with $h(r,\omega,x):=$*

$$r^{-k}\sup_{M>0}\frac{1}{M}\sum_{m=0}^{M-1}C_k^{\mathrm{var}}\left(\left(\psi_{\widetilde{\omega|m},u}F\right)_r,\psi_{\widetilde{\omega|m},u}\mathring{B}(x,ar)\right)1_{(R(x,1),R(x,0)]}\left(\frac{r}{\left|\psi'_{\widetilde{\omega|m},u}x\right|}\right).$$

(6) *The family of functions*

$$(\omega,x,r)\mapsto\frac{1}{\ln\frac{\epsilon_{\max}}{\epsilon}}\left[\rho^{k,\mathrm{var}}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(x,r,1)+\sum_{m\in\mathbb{N}}1\left\{r\left|\phi'_{\widetilde{\omega|m}}u\right|>\epsilon\right\}\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}(x,r,1)\right]\tag{3.0.1}$$

is uniformly $r^{-1}dr\,d\nu_{\mathrm{bi}}(\omega,x)$-integrable for all parameters $\epsilon\in(0,\epsilon_{\max}/2)$.

(7) *As above, except the integration variable x replaces u, including inside $\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}$.*

Intuitively, the assumption limits how much curvature the overlap sets may carry. More precisely, let τ, $\vartheta\in I^*$ be different shortest words such that the diameters of $\psi_{\widetilde{\omega|m},u}\phi_\tau F$ and $\psi_{\widetilde{\omega|m},u}\phi_\vartheta F$ are smaller than $ra^{-1}/\sqrt{2}$. Then to check (uniform) integrability of its $\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}$- or $r^{-k}C_k^{\mathrm{var}}\left(\left(\psi_{\widetilde{\omega|m},u}F\right)_r,\cdot\right)$-mass, only the intersection of $\psi_{\widetilde{\omega|m},u}\mathring{B}(x,ar)$ with the union of overlap sets $\left(\psi_{\widetilde{\omega|m},u}\phi_\tau F\right)_r\cap\left(\psi_{\widetilde{\omega|m},u}\phi_\vartheta F\right)_r$ of all such pairs τ, ϑ matters. The curvature on the non-overlap portion of $\psi_{\widetilde{\omega|m},u}\mathring{B}(x,ar)$ satisfies item (3) above, see [Zäh11, Theorem 4.1].

In case of self-similar fractals, further sufficient conditions are the Strong Curvature Bound Condition [Win11] and polyconvex parallel sets [Win08, Lemma 5.3.2]. Their advantage is they do not involve parallel sets widths r of all orders. (In our Zygmund space condition, $x\mapsto R(x,1)$ is bounded from below if the Strong Separation Condition holds.)

Not all self-conformal sets satisfy item (3) (see [RZ10]), hence the more general conditions.

Recall π (2.3.6) and f^{fib}_ω (2.1.10).

Theorem 3.3. *Let $k\in\{0,\ldots,d\}$, and F be a self-conformal set (Assumption 2.1). Suppose Assumptions 3.1 (only if $k\le d-1$) and 3.2 hold. Then for any continuous $f:V\times S^{d-1}\to\mathbb{R}$, the following limit exists and is finite:*

$$\begin{aligned}C_k^{frac}(F,f) &:= \lim_{\epsilon\searrow0}\frac{1}{\ln\epsilon_{\max}/\epsilon}\int_\epsilon^{\epsilon_{\max}}f(\underline{z})\,r^{D-k}dC_k(F_r,\underline{z})\frac{dr}{r}\\ &= \frac{D}{H_\nu}\int_{I^{\mathbb{N}}\times F}\int_0^\infty\pi^k_{\tilde{\omega}}\left(x,r,f^{\mathrm{fib}}_\omega\right)\frac{dr}{r}d\nu_{\mathrm{bi}}(\omega,x)\end{aligned}\tag{3.0.2}$$

$$=\frac{D}{H_\nu}\int_{I^{\mathbb{N}}\times F}\int_{R(x,1)}^{R(x,0)}\int_{\mathrm{nor}\widetilde{F_r}}\frac{r^{D-k}A\left(\left|\psi_{\tilde{\omega},x}z\right|,r\right)f^{\mathrm{fib}}_\omega(\underline{z})}{\int\left|\psi'_{\tilde{\omega},x}y\right|^DA\left(\left|\psi_{\tilde{\omega},x}y-\psi_{\tilde{\omega},x}z\right|,r\right)d\mu(y)}$$

$$dC_k\left((\psi_{\tilde{\omega},x}F)_r,\psi_{\tilde{\omega},x}\underline{z}\right)\frac{dr}{r}d\nu_{\mathrm{bi}}(\omega,x).$$

The first limit formula formally depends on u via $\psi_{\widetilde{\omega},u}$ inside $\pi^k_{\widetilde{\omega}}$. Its advantage is less objects depend on the integration variable x. The second formula is better adapted to the dynamical system variable (ω, x) and has u replaced with x. We conjecture the uniform integrability assumption (3.0.1) cannot be weakened, see Remark 4.4. The proof is postponed after Remark 4.4. Future work will simplify the formula for special cases.

For readers interested only in the Minkowski content, or in non-directional fractal curvature measures on $\mathbb{R}^d$ instead of $\mathbb{R}^d \times S^{d-1}$:

Corollary 3.4. *(Non-directional version) Assume the situation of the theorem, except that $f : \mathbb{R}^d \times S^{d-1}$ does not depend on its S^{d-1} coordinate, and positive reach (Assumption 3.1) is only needed for $k \le d-2$. Then the limit* (3.0.2) *exists. That limit formula holds, and $C^{frac}_k(F,\cdot)$ is proportional to μ, and $f^{\mathrm{fib}}_\omega = \int f\, d\mu$.*

In the self-similar setting, previous work used various definitions of A (Remark 4.12, [RZ10, Example 2.1.1]). We unify those approaches.

Corollary 3.5. *Let every ϕ_i be a similarity, in addition to Assumptions 2.1, 3.1, 3.2. (Note $\psi_{\dots} = \mathrm{id}$, $\left|\psi'_{\dots}\right| = 1$ therein.) Let A be as above, or the indicator function of any neighborhood net from* [RZ10, Example 2.1.1]*, or defined any other way that makes Lemma 4.13 true. Writing $\mathscr{H}^D$ for the D-dimensional Hausdorff measure, $\mathscr{H}_G$ for the Haar probability on $G := \overline{\left\langle \phi'_i : i \in I\right\rangle}_{O(d)}$, and c_R for the constants in* (2.3.2)*, we have*

$$C^{frac}_k(F,f) = \frac{D}{H_\nu} \int\limits_F \int\limits_{c_R \operatorname{dist}\left(x,\phi_{x|1} X^c\right)}^{c_R \operatorname{dist}(x,X^c)} \int\limits_{\operatorname{nor}\widetilde{F_r}} \frac{r^{D-k} A(|z-x|,r)\, f^{\mathrm{fib}}\left(\underline{z}\right)}{\int_F A\left(\left|y-z\right|,r\right)\, d\mathscr{H}^D(y)} dC_k\left(F_r,\underline{z}\right) \frac{dr}{r}\, d\mathscr{H}^D(x),$$

$$f^{\mathrm{fib}}(z,n) := \mathscr{H}^D(F)^{-1} \int\limits_F \int\limits_G f\left(x, gn\right)\, d\mathscr{H}^D(x)\, d\mathscr{H}_G\left(g\right).$$

Remark 3.6. Arguably, the fractal curvature-direction measures should instead be normalized as

$$\begin{aligned} \hat{C}^{\mathrm{frac}}_k(F,\cdot) &:= \binom{d-D_k}{d-k}^{-1} \pi^{\frac{D_k-k}{2}} \frac{\Gamma\left(\frac{d-D_k}{2}+1\right)}{\Gamma\left(\frac{d-k}{2}+1\right)} C^{\mathrm{frac}}_k(F,\cdot), \\ D_k &:= \min\left\{k, \overline{\dim}_{\mathrm{Minkowski}} F\right\}. \end{aligned} \tag{3.0.3}$$

The motivation is, embedding a set K into a larger ambient dimension d does not create geometric information. Stronger yet, Zähle conjectured the fractal curvatures $k > D$ never provide new information. This is proven in two cases: Fractal curvatures of an order k greater than the dimension of a "classical" set K simply repeat the highest available order of curvature (due to the Steiner formula),

$$\hat{C}^{\mathrm{frac}}_k(K,\cdot) = C_{\min\{k,\dim K\}}(K,\cdot) \text{ if } \operatorname{reach} K > 0.$$

For a general, closed, Lebesgue null set K, the fractal Lebesgue measure repeats the next-lower measure, [RW10, RW12]:

$$\hat{C}^{\mathrm{frac}}_d\left(K,\cdot\times S^{d-1}\right) = \hat{C}^{\mathrm{frac}}_{d-1}\left(K,\cdot\times S^{d-1}\right) \text{ if } C_d(K) = 0. \tag{3.0.4}$$

4. Proofs

4.1. **Exposing the dynamical system.** The purpose of the next proposition is to rephrase the problem, of taking Cesaro limits of volume or curvature of parallel sets, into the language of the ergodic shift dynamical system (F,σ,ν). The geometric intuition is to study the curvature in a small ball around $x\in F$ as the parallel set width r decreases. The conformal measure μ of this ball balances out the rescaling factor. Instead of actual balls, we will use a tent function A supported by the ball (2.3.1). We introduce the extra integrand $1=\int A\,d\mu/\int A\,d\mu$ into the curvature and use Fubini to make the "dynamical" measure μ accessible. The analytic intuition is to convolute the curvature measure with an integrable bump function A. If we had a sensible (harmonic analysis) group structure on the fractal, it would preserve the metric. So we make the bump $x\mapsto A(|x-z|,r)$ depend only on pairwise distances $|x-z|$. Lacking a global group, we normalize the bump mass locally at each midpoint x.

This leads us to study the dynamical system's integrands ρ, π for the rest of this paper. Recall from (2.3.4),

$$\rho_{(s,t]}^{k,\pm}(x,r,f):=1_{(s,t]}(r)\int\limits_{\operatorname{nor}\widetilde{F_r}} f\left(\underline{z}\right)\frac{r^{D-k}A(|x-z|,r)}{\int A\left(\left|y-z\right|,r\right)d\mu\left(y\right)}\,dC_k^{\pm}\left(F_r,\underline{z}\right).$$

Remark 4.1. Borel measurability of ρ and π as a function of r can be proved as in [RZ10, Lemma 2.3.1]. The mapping $r\mapsto C_k(F_r,\cdot)$ is weak*-continuous at almost every r (see [Zäh11, Corollary 2.3.5]). The non-osculating Assumption 3.1 (2) is used here. Since $C_k\left(\left(\psi F\right)_r,\cdot\right)$ agrees with a (limit of) pullback of some $C_k\left(\left(\psi F\right)_{\mathrm{const}\,r},\cdot\right)$ on sets we integrate over, we will see it is automatically measurable.

Proposition 4.2. *Let $k\in\{0,\dots,d\}$, $\epsilon>0$, and $f:V\times S^{d-1}\to[0,\infty)$ be any nonnegative, measurable function. Suppose Assumption 2.1 governs our self-conformal set F. Its parallel sets F_r shall be regular if $k\le d-1$ (but only $k\le d-2$ for $f:V\to[0,\infty)$), Assumption 3.1. Then we have*

(4.1.1)
$$\frac{1}{\ln\epsilon_{\max}/\epsilon}\int\limits_{\epsilon}^{\epsilon_{\max}} f\left(\underline{z}\right)r^{D-k}dC_k^{\pm}\left(F_r,\underline{z}\right)\frac{dr}{r}=\int\limits_F\int\limits_0^{\infty}\frac{1}{\ln\epsilon_{\max}/\epsilon}\rho_{(\epsilon,\epsilon_{\max}]}^{(k,\pm)}(x,r,f)\,\frac{dr}{r}d\mu(x).$$

Proof. At each point $z\in F_r$ we may introduce an arbitrary factor

$$1=\frac{\int A(|x-z|,r)\,d\mu(x)}{\int A\left(\left|y-z\right|,r\right)d\mu\left(y\right)}. \tag{4.1.2}$$

Lemma 4.13 (1), (3) makes sure $\int A\,d\mu>0$ and $A\ge0$. Next, we may apply Fubini's theorem because we are dealing with positive integrands and finite measures. If $\epsilon<r\le\epsilon_{\max}$:

$$r^{D-k}\int\limits_{\operatorname{nor}\widetilde{F_r}} f\left(\underline{z}\right)\,dC_k^{\pm}\left(F_r,\underline{z}\right)=r^{D-k}\int\limits_{\operatorname{nor}\widetilde{F_r}} f\left(\underline{z}\right)\,\frac{\int A(|x-z|,r)\,d\mu(x)}{\int A\left(\left|y-z\right|,r\right)d\mu\left(y\right)}\,dC_k^{\pm}\left(F_r,\underline{z}\right)$$
$$=\int\limits_J\int\limits_{\operatorname{nor}\widetilde{F_r}} f\left(\underline{z}\right)\frac{r^{D-k}A(|x-z|,r)}{\int A\left(\left|y-z\right|,r\right)d\mu\left(y\right)}\,dC_k^{\pm}\left(F_r,\underline{z}\right)d\mu(x)=\int\limits_J\rho_{(\epsilon,\epsilon_{\max}]}^{(k,\pm)}(x,r,f)\,d\mu(x).$$

Finally, we integrate over $r^{-1}dr$ and apply Fubini again. $\square$

4.2. **Dualizing the dynamics to apply the Birkhoff theorem and leave the distortion behind.** Here we tell the story arc of this paper and treat the dynamics. The curvature Cesaro average is split into chunks mapped onto each other up to distortion of the underlying fractal. The local geometric covariance matches up only their curvature measures. To compensate, the dynamical system pulls back the test function. Asymptotics of the distortion exist only if the ordering of the conformal maps ϕ_i is reversed. The Perron-Frobenius operator achieves this, but requires the two-sided extension $I^{\mathbb{N}} \times F$ of the code space. (Intuitively, summing over all possible n-letter words is the same as summing over reversed words.) It also separates the distortion inside the measure from the dynamics map inside the test function. A vector (Kojima) Toeplitz theorem moves the curvature measures out of the Cesaro average. The Birkhoff theorem for the left shift fits the Cesaro-averaged, pulled-back chunks of test function after compensating the following: The curvature lives on a parallel set "beside" the dynamical system F. This selects a different element from the covariance group element that transforms the normal directions.

This work essentially reduces the self-conformal case to similarities, at the price of extra distortion. The distortion and the geometry packaged in ρ, π will later be treated point-wise using the right shift $\sigma_{\mathrm{bi}}^{-1}$. From a dynamics point of view, the different roles of left and right shift explain why the conformal measure μ_{bi} integrates the test function in the limiting C_k^{frac}.

Recall ρ from (2.3.4), π from (2.3.5) and (2.3.6).

Proposition 4.3. *(core of proof) Assume $k \in \{0, \ldots, d\}$, $f : V \times S^{d-1} \to \mathbb{R}$ continuous and F is self-conformal (Assumption 2.1). Let its parallel sets F_r be regular if $k < d$ (only $k < d-1$ if $f : V \to \mathbb{R}$) (positive reach Assumption 3.1). Let the expression*

$$\frac{1}{\ln \frac{\epsilon_{\max}}{\epsilon}} \left[\rho^{k,\mathrm{var}}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(x,r,1) + \sum_{n\in\mathbb{N}} 1\left\{ r \left| \phi'_{\widetilde{\omega|n}} u \right| > \epsilon \right\} \pi^{k,\mathrm{var}}_{\widetilde{\omega|n}}(x,r,1) \right] \tag{4.2.1}$$

be ϵ-uniformly $r^{-1}dr\, d\mu_{\mathrm{bi}}(\omega,x)$-integrable. Then for any continuous $f : V \times S^{d-1} \to \mathbb{R}$, the limit

$$\begin{aligned} &\lim_{\epsilon\searrow 0} \frac{1}{\ln \epsilon_{\max}/\epsilon} \int_{\epsilon}^{\epsilon_{\max}} r^{D-k} \int_{\mathbb{R}^d \times S^{d-1}} f(\underline{z})\, dC_k(F_r, \underline{z}) \frac{dr}{r} \\ &= \frac{D}{H_\nu} \int_{I^{\mathbb{N}} \times F} \int_0^\infty \pi^k_{\tilde{\omega}}\left(x, r, f^{\mathrm{fib}}_\omega\right) \frac{dr}{r} d\nu_{\mathrm{bi}}(\omega, x) \end{aligned} \tag{4.2.2}$$

exists.

Proof. Due to linearity, we may assume $0 \le f \le 1$.

First, we will rewrite (4.2.2) for any fixed ϵ. To prevent that the $r^{-1}dr$-integral evaluates to $\infty - \infty$, we will work with C_k^+ and C_k^- separately at first. Proposition 4.2 below uses Fubini's theorem to makes μ appear in the next line,

$$L := \frac{1}{\ln \frac{\epsilon_{\max}}{\epsilon}} \int_{\epsilon}^{\epsilon_{\max}} \int_{\mathrm{nor}\,\widetilde{F_r}} f(\underline{z})\, r^{D-k} dC_k^{\pm}(F_r, \underline{z}) \frac{dr}{r} = \frac{1}{\ln \frac{\epsilon_{\max}}{\epsilon}} \int_F \int_0^\infty \rho^{k,\pm}_{(\epsilon,\epsilon_{\max}]}(x, r, f) \frac{dr}{r} d\mu(x).$$

The sequence sequence $n \mapsto R(x,n)$ is strictly falling (Lemma 4.11 (3) (4)). We can split the dr integral along it without reversing the new bounds of any chunk of the integral. The indicator function corresponding to the bounds $\int_{R(x,n+1)}^{R(x,n)}$ is pulled into the subscript of ρ, see (2.3.4):

$$L = \int_F \int_0^\infty \frac{\rho^{k,\pm}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(x,r,f)}{\ln\frac{\epsilon_{\max}}{\epsilon}} + \sum_{n\in\mathbb{N}} \frac{1\{r>\epsilon\}}{\ln\frac{\epsilon_{\max}}{\epsilon}} \rho^{k,\pm}_{(R(x,n+1),R(x,n)]}(x,r,f)\,\frac{dr}{r}\,d\mu(x).$$

Next, Lemma 4.14 below asserts $\rho^{k,\pm}_{(R(x,n+1),R(x,n)]}(x,r,f) = \pi^{k,\pm}_{x|n}\left(\sigma^n x, r\left|\phi'_{x|n}u\right|^{-1}, f\circ\phi_{x|n}\right)$. The geometric meaning is we preimage it under $\phi_{x|n}$ and use the covariance and locality properties of curvature. Hence,

$$L = \int_F \int_0^\infty \cdots + \sum_{n\in\mathbb{N}} \frac{1\{r>\epsilon\}}{\ln\frac{\epsilon_{\max}}{\epsilon}} \pi^{k,\pm}_{x|n}\left(\sigma^n x, \frac{r}{\left|\phi'_{x|n}u\right|}, f\circ\phi_{x|n}\right)\frac{dr}{r}\,d\mu(x).$$

Temporarily, we pull the formally infinite sum out of both integrals. Each summand integral exists due to positivity (or uniform integrability once proved),

$$L = \int_F \int_0^\infty \cdots + \sum_{n\in\mathbb{N}} \int_F \int_0^\infty \frac{1\{r>\epsilon\}}{\ln\frac{\epsilon_{\max}}{\epsilon}} \pi^{k,\pm}_{x|n}\left(\sigma^n x, \frac{r}{\left|\phi'_{x|n}u\right|}, f\circ\phi_{x|n}\right)\frac{dr}{r}\,d\mu(x).$$

By passing to the two-sided code space and to the invariant measure $\nu_{\text{bi}} = p\mu_{\text{bi}}$, we can apply the shift operator, $\left(\sigma^n\omega, \phi_{\widetilde{\omega|n}}x\right) = \sigma^{-n}_{\text{bi}}(\omega,x)$, (2.1.3). It consistently replaces x with $\phi_{\widetilde{\omega|n}}x$. But left and right shifts are inverse operations. Thus it replaces the original $\sigma^n x$ with $\sigma^n\phi_{\widetilde{\omega|n}}x = x$ and $x|n$ with $\left(\phi_{\widetilde{\omega|n}}x\right)|n = \widetilde{\omega|n}$:

$$L = \int_F \int_0^\infty \cdots + \sum_{n\in\mathbb{N}} \int_{I^{\mathbb{N}}\times F} \int_0^\infty \frac{1\{r>\epsilon\}}{p\left(\phi_{\widetilde{\omega|n}}x\right)\ln\frac{\epsilon_{\max}}{\epsilon}} \pi^{k,\pm}_{\widetilde{\omega|n}}\left(x, \frac{r}{\left|\phi'_{\widetilde{\omega|n}}u\right|}, f\circ\phi_{\widetilde{\omega|n}}\right)\frac{dr}{r}\,d\nu_{\text{bi}}(\omega,x).$$

We substitute $r/\left|\phi'_{\widetilde{\omega|n}}u\right| \mapsto r$ in the dr integral and put the sum back inside,

$$L = \int_{I^{\mathbb{N}}\times F} \int_0^\infty \left[\cdots + \sum_{n\in\mathbb{N}} \frac{1\left\{\left|\phi'_{\widetilde{\omega|n}}u\right| > \frac{\epsilon}{r}\right\}}{p\left(\phi_{\widetilde{\omega|n}}x\right)\ln\frac{\epsilon_{\max}}{\epsilon}} \pi^{k,\pm}_{\widetilde{\omega|n}}\left(x, r, f\circ\phi_{\widetilde{\omega|n}}\right)\right]\frac{dr}{r}\,d\nu_{\text{bi}}(\omega,x).$$

Because we want to move the ϵ-limit inside both integrals, we must assume the last line is uniformly integrable. The bounds $K^{-1}\le p\le K$ and $\left\|f\circ\phi_{\widetilde{\omega|n}}\right\|\le 1$ simplify that to our stated assumption (3.0.1).

Uniform integrability of L implies integrability. So we can subtract our chain of equations for C_k^+ and C_k^-, i.e., $\pi^{k,+}_{\widetilde{\omega|n}} - \pi^{k,-}_{\widetilde{\omega|n}}$. Then we draw the limit $\epsilon\to 0$ into the integrals.

We have proved, assuming the limit inside the double integral exists:

(4.2.3)

$$\lim_{\epsilon\to 0}\frac{1}{\ln\frac{\epsilon_{\max}}{\epsilon}}\int_{\epsilon}^{\epsilon_{\max}}\int_{\operatorname{nor}\widetilde{F_r}} f\left(\underline{z}\right)\, r^{D-k}\,dC_k\left(F_r,\underline{z}\right)\frac{dr}{r}=\int_{I^{\mathbb{N}}\times F}\int_0^{\infty}\lim_{\epsilon\to 0}\left[\frac{\rho^{k,\pm}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}\left(x,r,f\right)}{\ln\frac{\epsilon_{\max}}{\epsilon}\,p\left(x\right)}\right.$$
$$\left.+\frac{1}{\ln\frac{\epsilon_{\max}}{\epsilon}}\sum_{n\in\mathbb{N}}1\left\{\left|\phi'_{\widetilde{\omega|n}}u\right|>\frac{\epsilon}{r}\right\}\pi^k_{\widetilde{\omega|n}}\left(x,r,\frac{f\circ\phi_{\widetilde{\omega|n}}}{p\left(\phi_{\widetilde{\omega|n}}x\right)}\right)\right]\frac{dr}{r}\,d\nu_{\mathrm{bi}}(\omega,x).$$

The rest of this proof will show the integrand limit (4.2.3) exists. Clearly, $\rho^{k,\pm}_{\dots}$ can be ignored,

$$\lim_{\epsilon\to 0}\left(\ln\frac{\epsilon_{\max}}{\epsilon}\right)^{-1}\frac{\rho^{k}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}\left(x,r,f\right)}{p\left(x\right)}=0.$$

The Markov time $N\left(r/\epsilon\right):=\max\left\{n\in\mathbb{N}:\left|\phi'_{\widetilde{\omega|n}}u\right|>\frac{\epsilon}{r}\right\}$ will help reinterpret the sum as Cesaro average. The integrand of (4.2.3) becomes

$$\lim_{\epsilon\to 0}\left[0+\frac{N\left(r/\epsilon\right)}{\ln\epsilon_{\max}/\epsilon}\frac{1}{N\left(\frac{r}{\epsilon}\right)}\sum_{n\le N\left(\frac{r}{\epsilon}\right)}\pi^k_{\widetilde{\omega|n}}\left(x,r,\frac{f\circ\phi_{\widetilde{\omega|n}}}{p\left(\phi_{\widetilde{\omega|n}}x\right)}\right)\right].$$

Lemma 4.6 below allows us to replace the following subexpression with its limit,

$$\frac{N\left(r/\epsilon\right)}{\ln\epsilon_{\max}/\epsilon}\underset{\epsilon\to 0}{\longrightarrow}\frac{D}{H_\nu}.$$

That eliminated the only ϵ outside $N\left(r/\epsilon\right)$. Recall $\left|\phi'_{\widetilde{\omega|n}}u\right|\le s^n_{\max}$, $s_{\max}<1$. Thus $\epsilon\to 0$ implies $N\left(r/\epsilon\right)\to\infty$. We replace $\lim_{\epsilon\to 0}$ with $\lim_{N\to\infty}$. Inside the integrals in (4.2.3), that leaves

$$0+\frac{D}{H_\nu}\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\pi^k_{\widetilde{\omega|n}}\left(x,r,\frac{f\circ\phi_{\widetilde{\omega|n}}}{p\left(\phi_{\widetilde{\omega|n}}x\right)}\right).$$

If α_n is a weakly converging sequence of signed measures and b_n a sequence of equicontinuous functions, then the Cesaro average $\frac{1}{N}\sum_{n\le N}\alpha_\infty\left(b_n\right)$ has the same limiting behavior as $\frac{1}{N}\sum_{n\le N}\alpha_n\left(b_n\right)$. In other words, any converging measure α_n may be replaced by its limit α_∞, see Lemma 5.1 below. Let us verify the assumptions: Lemma 4.16 below provides the geometric justification that $\alpha_n\left(\cdot\right):=\pi^k_{\widetilde{\omega|n}}\left(x,r,\cdot\right)$ does converge. The expression

$$f^{\mathrm{fib}}_\omega\left(\underline{z}\right)=\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\frac{f\circ\phi_{\widetilde{\omega|n}}\left(\underline{z}\right)}{p\left(\phi_{\widetilde{\omega|n}}x\right)}$$

will be $\lim\frac{1}{N}\sum_{n\le N}b_n$. It converges uniformly in $\underline{z}\in\overline{V}\times S^{d-1}$ for almost all (ω,x): Lemma 4.5 below provides the reduction to the individual ergodic theorem (and a formula). The family of all ϕ-translates of $\underline{z}\mapsto f\left(\underline{z}\right)$ is equicontinuous. So we may replace $\pi^k_{\widetilde{\omega|n}}$ with $\pi^k_{\tilde{\omega}}$ in (4.2.3). We have achieved our goal to show the integrand limit exists:

$$\lim_{\epsilon\to 0}\frac{\rho_{(\dots,\epsilon_{\max}]}}{\ln\frac{\epsilon_{\max}}{\epsilon}p}+\frac{1}{N}\frac{N}{\ln\epsilon_{\max}/\epsilon}\sum_{n\le N}\pi^k_{\widetilde{\omega|n}}\left(x,r,\frac{f\circ\phi_{\widetilde{\omega|n}}}{p\left(\phi_{\widetilde{\omega|n}}x\right)}\right)=\frac{D}{H_\nu}\pi^k_{\tilde{\omega}}\left(x,r,f^{\mathrm{fib}}_\omega\right).$$

□

Remark 4.4. A converse shows the uniform integrability condition (4.2.1) likely cannot be improved. Without it, the same proof yields for continuous $f \geq 0$,

$$\liminf_{\epsilon\to 0}\frac{1}{\ln\frac{\epsilon_{\max}}{\epsilon}}\int_F\int_0^\infty\left|\rho^k_{(\epsilon,\epsilon_{\max}]}(x,r,f)\right|\frac{dr}{r}d\mu(x)\geq\frac{D}{H_\nu}\int_{I^{\mathbb{N}}\times F}\int_0^\infty\left|\pi^k_{\widetilde{\omega}}\left(x,r,f^{\mathrm{fib}}_\omega\right)\right|\frac{dr}{r}d\nu_{\mathrm{bi}}(\omega,x).$$

Both equality holds and the lower limit exists as a proper, finite limit, if and only if (4.2.1) after replacing ρ^{var} by $\left|\rho\right|$ and π^{var} by $|\pi|$ is uniformly integrable. This follows from a converse to Fatou's lemma ([Doo94, Chapters 6.8 and 6.18]).

Assume we may replace C_k with $C_k^\pm$ in the continuity assertion of Fact 4.15 below when proving (4.6.1) (i.e. $\lim_n \alpha_n$). Then the last inequality improves to

(4.2.4)
$$\liminf_{\epsilon\to 0}\frac{1}{\ln\frac{\epsilon_{\max}}{\epsilon}}\int_\epsilon^{\epsilon_{\max}} r^{D-k}\int_{\mathrm{nor}\widetilde{F_r}} f\left(\underline{z}\right)\,dC_k^\pm\left(F_r,\underline{z}\right)\frac{dr}{r}\geq\frac{D}{H_\nu}\int_{I^{\mathbb{N}}\times F}\int_0^\infty \pi^{k,\pm}_{\widetilde{\omega}}\left(x,r,f^{\mathrm{fib}}_\omega\right)\frac{dr}{r}d\nu_{\mathrm{bi}}(\omega,x).$$

Both equality holds and the lower limit exists as a proper, finite limit, if and only if (4.2.1) after replacing ρ^{var} by $\rho^\pm$ and π^{var} by $\pi^\pm$ is uniformly integrable. (This was proved for self-similar F in [BZ11, Proposition 3.12].)

Proof. (of Theorem 3.3) Lemmas 4.17, 4.18 derive uniform integrability with u (3.0.1) from the other conditions with u in Assumption 3.2. Then Proposition 4.3 above guarantees convergence to the first limit formula (with u). In a second run, we replace the constant u by the integration variable x in all the proofs (except in Lemma 4.9, where it plays a different role). Again, the other conditions without u imply uniform integrability without u and convergence to the second formula (without u). It remains to show both formulas are equal.

Write $\pi^{k,u}_{\widetilde{\omega}}$ for $\pi^k_{\widetilde{\omega}}$, and $\pi^{k,x}_{\widetilde{\omega}}$ when all instances of u in its definition (2.3.6) are replaced with x. This is the second limit formula's integrand in the theorem. Define $\pi^{k,u}_{\widetilde{\omega|m}}$, $\pi^{k,x}_{\widetilde{\omega|m}}$ accordingly for finite words $\widetilde{\omega|m}$. Once as stated and once with u replaced, (4.5.4) implies

$$\pi^{k,u}_{\widetilde{\omega|m}}(x,r,f)=\rho^k_{(\dots]}\left(\phi_{\widetilde{\omega|m}}x,r\left|\phi'_{\widetilde{\omega|m}}u\right|,f\circ\phi^{-1}_{\widetilde{\omega|m}}\right)=\pi^{k,x}_{\widetilde{\omega|m}}\left(x,r/\left|\psi'_{\widetilde{\omega|m},u}x\right|,f\right).$$

Then as $m\to\infty$, Lemma 4.16 gives $\pi^{k,u}_{\widetilde{\omega}}(x,r,f)=\pi^{k,x}_{\widetilde{\omega}}\left(x,r/\left|\psi'_{\widetilde{\omega},u}x\right|,f\right)$. This and the coordinate transformation $(\omega,x,r)\mapsto\left(\omega,x,r/\left|\psi'_{\widetilde{\omega},u}x\right|\right)$ turn both limit formulas into each other. □

The point (η,u), the Brin group $G_{\eta u}$, and f^{fib}_ω were defined in (2.1.9). Recall

$$f^{\mathrm{fib}}_\omega(z,n):=\int_{I^{\mathbb{N}}\times F}\int_{G_{\eta u}} f\left(\hat{y},\left(\psi'_{\widetilde{\eta},u}\hat{y}\right)^{\mathrm{orth}-1}\hat{g}\left(\psi'_{\widetilde{\omega},u}z\right)^{\mathrm{orth}}n\right)d\mathscr{H}_G\left(\hat{g}\right)d\mu\left(\hat{y}\right)$$

does not depend on (η,u).

To readers not interested in directed curvature, the next lemma merely recalls Birkhoff's theorem.

Although the dynamical system map $\phi_{\widetilde{\omega|m}}$ is contractive in $\overline{V}$, the limit f_ω^{fib} can depend on $z \in \overline{V}$ via the orbit of $n \in S^{d-1}$.

Lemma 4.5. *For almost all (ω, x), the expression*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m\le M} \frac{f\circ\phi_{\widetilde{\omega|m}}(z,n)}{p\left(\phi_{\widetilde{\omega|m}}x\right)} = f_\omega^{\text{fib}}(z,n), \tag{4.2.5}$$

converges uniformly in (z,n), i.e. w.r.t the norm on $\overline{V}\times S^{d-1}$. It can be interpreted as the $\underline{\nu_{\text{bi}}}$-conditional expectation of the function $a_{(\omega,x,z,n)}$ on $\underline{\sigma_{\text{bi}}}$-orbits at the point (ω, x, id), where

$$a_{(\tau,y,z,n)}(\omega,x,g) := p(x)^{-1} f\left(x, g\left(\psi'_{\widetilde{\tau},y} z\right)^{\text{orth}} n\right).$$

The limit simplifies to a constant in case f does not depend on the normal variable $n \in S^{d-1}$,

$$f_\omega^{\text{fib}}(z,n) = \int f\, d\mu.$$

Proof. Recall the extended dynamical system $\left(I^{\mathbb{N}}\times F\times O(d), \underline{\sigma_{\text{bi}}}, \underline{\nu_{\text{bi}}}\right)$, $\underline{\nu_{\text{bi}}} = \nu_{\text{bi}}\otimes\mathscr{H}_{O(d)}$,

$$\underline{\sigma_{\text{bi}}}^{-m}(\omega,x,g) = \left(\sigma^m\omega, \phi_{\widetilde{\omega|m}}x, \left(\phi'_{\widetilde{\omega|m}}x\right)^{\text{orth}} g\right).$$

We will rewrite $f\circ\phi_{\widetilde{\omega|m}}$ asymptotically in terms of this system. The base point z can be moved to x at the price of an additional ψ (defined in (2.1.7)):

$$\begin{aligned} f\circ\phi_{\widetilde{\omega|m}}(z,n) &= f\left(\phi_{\widetilde{\omega|m}}z, \left(\phi'_{\widetilde{\omega|m}}z\right)^{\text{orth}} n\right) \\ &= f\left(\phi_{\widetilde{\omega|m}}z, \left(\phi'_{\widetilde{\omega|m}}x\right)^{\text{orth}} \left(\psi'_{\widetilde{\omega|m},x}z\right)^{\text{orth}} n\right). \end{aligned}$$

The asymptotic approximation of $\frac{f}{p}\circ\phi_{\widetilde{\omega|m}}$ will be later obtained from

$$a_{(\tau,y,z,n)}\circ\underline{\sigma_{\text{bi}}}^{-m}(\omega,x,g) = \frac{f\left(\phi_{\widetilde{\omega|m}}x, \left(\phi'_{\widetilde{\omega|m}}x\right)^{\text{orth}} g\left(\psi'_{\widetilde{\tau},y}z\right)^{\text{orth}} n\right)}{p\left(\phi_{\widetilde{\omega|m}}x\right)} \tag{4.2.6}$$

by setting $g := \text{id}$, $(\tau, y) := (\omega, x)$. Indeed, the estimates $\left|\phi_{\widetilde{\omega|m}}x - \phi_{\widetilde{\omega|m}}z\right| \le s_{\max}^m \operatorname{diam} V$ and $\left|\left(\psi'_{\widetilde{\omega},x}z\right) - \left(\psi'_{\widetilde{\omega|m},x}z\right)\right| \le s_{\max}^{m\gamma}c_\psi$ depend only on m. Since f is uniformly continuous, this implies uniform convergence

$$\left|\frac{1}{M}\sum_{m\le M}\left(\frac{f\circ\phi_{\widetilde{\omega|m}}(z,n)}{p\left(\phi_{\widetilde{\omega|m}}x\right)} - a_{(\omega,x,z,n)}\circ\underline{\sigma_{\text{bi}}}^{-m}(\omega,x,\text{id})\right)\right| \underset{M\to\infty}{\longrightarrow} 0, \tag{4.2.7}$$

i.e., at a rate that does not depend on ω, x, z, n.

Next, we will apply the Birkhoff ergodic theorem to a. Let $\mathscr{D}$ be a countable, dense set of $(\tau, y, z, n) \in I^{\mathbb{N}}\times F\times\overline{V}\times S^{d-1}$. On each element of $\mathscr{D}$, the limit

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m\le M} a_{(\tau,y,z,n)}\circ\underline{\sigma_{\text{bi}}}^{-m}(\omega,x,g) \tag{4.2.8}$$

exists for almost every (ω, x, g). Lemma 4.9 provides a formula (a $\underline{\nu_{\text{bi}}}$-conditional expectation of $a_{(\tau,y,z,n)}$ on $\underline{\sigma_{\text{bi}}}$-invariant sets). Once we know it converges uniformly in (τ, y, z, n) and g at each fixed (ω, x), we will be allowed to set $(\tau, y) = (\omega, x)$ and $g = \text{id}$. Then (4.2.7) and (4.2.8) will yield the assertion.

The auxiliary function

$$(\tau, y, z, n, g) \mapsto g\left(\psi'_{\tilde{\tau},y} z\right)^{\text{orth}} n$$

is uniformly continuous, see Proposition 4.8 (1) for $\psi'_{\tilde{\tau},y} z$. We insert this into the explicit formulas (4.2.6) for $a \circ \underline{\sigma_{\text{bi}}}^{-m}$ and (2.1.10) for f_ω^{fib}. So both (4.2.8) and f_ω^{fib} are uniformly equicontinuous w.r.t. (τ, y, z, n) and g at each fixed (ω, x). Equality extends to the closures $(\tau, y, z, n) \in \overline{\mathscr{D}}$, $g \in O(d)$. The Arzela-Ascoli theorem improves point-wise convergence on $\overline{\mathscr{D}} \times O(d)$ to the uniform convergence used above. We have shown

$$\frac{1}{M} \sum_{m \le M} \frac{f \circ \phi_{\widetilde{\omega|m}}(z, n)}{p\left(\phi_{\widetilde{\omega|m}} x\right)} \xrightarrow[M]{}$$
$$\int_{I^{\mathbb{N}} \times F} \int_{G_{\eta u}} a_{(\omega,x,z,n)}\left(\hat{\vartheta}, \hat{y}, \left(\psi'_{\tilde{\eta},u} \hat{y}\right)^{\text{orth}-1} \hat{h} \left(\psi'_{\tilde{\eta},u} x\right)^{\text{orth}} \text{id}\right) d\mathscr{H}_G\left(\hat{h}\right) d\nu_{\text{bi}}\left(\hat{\vartheta}, \hat{y}\right).$$

Finally, $p\left(\phi_{\widetilde{\omega|m}} x\right)^{-1} - p\left(\phi_{\widetilde{\omega|m}} u\right)^{-1}$ asymptotically vanishes for any $x, u \in V$ because $\phi_{\widetilde{\omega|m}}$ contracts and p is Hölder continuous. So the left side the same for almost all values of x. Since the right side is continuous, it does not depend on the choice of x at all. Set $x := u$ and simplify $\psi'_{\tilde{\eta},u} u = \text{id}$, $\nu_{\text{bi}} = \nu$ lacking $\hat{\vartheta}$, $\nu / p = \mu$. □

Recall (2.3.3): $H_\nu := -D \int_F \ln \left|\phi'_{x_1} \sigma x\right| d\nu(x)$.

Lemma 4.6. *Writing*

$$N\left(\omega, u, \frac{\epsilon}{r}\right) := \max\left\{n \in \mathbb{N} : \left|\phi'_{\widetilde{\omega|n}} u\right| > \frac{\epsilon}{r}\right\},$$

we have

$$\lim_{\epsilon \to 0} \frac{\ln \epsilon_{\max}/\epsilon}{N(\omega, u, \epsilon/r)} = \frac{H_\nu}{D}$$

for ν_{bi}*-almost all* ω *and all* $u \in F$.

Proof. See the first half of the proof of [Pat04, Lemma 4.7] for convergence and identification of the limit, and e.g. [PU10, Thm. 1.9.7] for the entropy interpretation. Bounded distortion extends it to all $u \in F$. □

4.3. **Time-averaged distance, distortion, and ergodic fibres.** The iterated function system maps F onto some smaller building block $\phi_{\omega|n} F$ of the fractal under the non-linear contraction $\phi_{\omega_n} \circ \cdots \circ \phi_{\omega_1} = \phi_{\omega|n}$. But curvature and parallel sets transform nicely under a similarity $\phi'_{\omega|n} x$. To compensate, the (non-contracting) distortion $\psi_{\omega|n,x}$ magnifies by one and contracts by the other map. It usually does not converges as $n \to \infty$ for an infinite sequence $\omega \in I^{\mathbb{N}}$. This is why curvature densities in the sense of [RZ10, BZ11] do not exist. Following [Zäh01], the solution is to reverse the sub-words, $\widetilde{\omega|n} := \omega_n \omega_{n-1} \ldots \omega_1$. The two-sided right shift dynamical system inverts the sub-words: $\sigma_{\text{bi}}^{-n}(\omega, x) = \left(\sigma^n \omega, \phi_{\widetilde{\omega|n}} x\right)$. Hölder techniques will show $\psi_{\widetilde{\omega|n},x}$ does converge. Our $\lim_n \left|\psi_{\widetilde{\omega|n},x} y\right|$ is equal to Zähle's time-averaged distance function $\text{dist}_\omega(x, y)$.

It is well-known the two-sided shift dynamical system $I^{\mathbb{N}} \times F$ has an ergodic Gibbs measure ν_{bi}. But the directional nature of our curvature requires a further extension by the orthogonal group. We provide an explicit formula for its ergodic fibre decomposition. A key insight, the distortion derivative ψ' arises as a skewing factor, by which the fibres differ from a product measure. (Readers interested in curvature on $\mathbb{R}^d$ without the normals can ignore ψ' and Lemma 4.9.)

Recall (2.1.7),

$$\psi_{\widetilde{\omega|n},x} y \quad := \quad \left(\phi'_{\widetilde{\omega|n}} x\right)^{-1} \left(\phi_{\widetilde{\omega|n}} y - \phi_{\widetilde{\omega|n}} x\right),$$

Notice

$$K^{-1} \le \left|\psi'_{\tau,x} y\right| = \frac{\left|\phi'_\tau y\right|}{\left|\phi'_\tau x\right|} \le K \tag{4.3.1}$$

and $\psi_{\omega|n,x} y = \left(\psi'_{\omega|n,u} x\right)^{-1} \left(\psi_{\omega|n,u} y - \psi_{\omega|n,u} x\right)$.

Fact 4.7. *A sequence a_n is said to convergence at the rate $c_\psi s^n$ if $\sup_k |a_n - a_{n+k}| \le c_\psi s^n$ for all n. Note that if a_n and b_n converge at the rates $c_a s^n$ and $c_b s^n$, then the sequences $a_n b_n$ and $1/a_n$ converge at the rates $c_{ab} s^n$ and $c_{1/a} s^n$ respectively, with $c_{ab} \le |a_1| c_b + |b_1| c_a + 2 c_a c_b s$ and $c_{1/a} \le c_a / \inf_n |a_n|^2$.*

Proposition 4.8. *(Distortion converges) For all $x, y \in \bar{V}$ and all codes $\omega \in I^{\mathbb{N}}$,*

(1) *Both*

$$\begin{aligned} \psi_{\tilde{\omega},x} y \quad &:= \quad \lim_{n\to\infty} \psi_{\widetilde{\omega|n},x} y, \\ \psi'_{\tilde{\omega},x} y \quad &= \quad \lim_{n\to\infty} \psi'_{\widetilde{\omega|n},x} y \end{aligned}$$

converge at the universal rate $c_\psi s_{\max}^{n\gamma}$. The constant $0 < c_\psi < \infty$ does not depend on x, y, ω. Thus both limiting functions are uniformly continuous in ω, x, y. (The derivative of the limit $\psi_{\tilde{\omega},x}$ agrees with the limit of the derivative $\psi'_{\widetilde{\omega|n},x}$.)

(2) *There is a constant K_ψ such that*

$$K_\psi^{-1} \le \frac{\left|\psi_{\widetilde{\omega|n},x} y\right|}{\left|x - y\right|} \le K_\psi.$$

(3) $\psi_{\omega|n,x} \circ \phi^{-1}_{\omega|n}$ *is a similarity with Lipschitz constant* $\left|\phi'_{\omega|n} x\right|^{-1}$.

(4) $\psi_{\widetilde{\omega|n},x} x = 0$ *and* $\psi'_{\widetilde{\omega|n},x} x = \mathrm{id}$.

Proof. For convenience, set $\tau := \widetilde{\omega|n}$.

For the second assertion, note [Pat97, Lemma 2] states

$$\text{const} \le \frac{\left|\phi_{\widetilde{\omega|n}} y - \phi_{\widetilde{\omega|n}} x\right|}{\left\|\phi'_{\widetilde{\omega|n}}\right\| \left|x - y\right|} \le \text{const}.$$

(Here, we made use of the $C^{1+\gamma}$ conformal extension of ϕ_i to $\overline{V_{MU}}$, subject to (2.1.6).) Then $\left|\left(\phi'_{\widetilde{\omega|n}} x\right)^{-1} \left(\phi_{\widetilde{\omega|n}} y - \phi_{\widetilde{\omega|n}} x\right)\right| = \left|\phi_{\widetilde{\omega|n}} y - \phi_{\widetilde{\omega|n}} x\right| / \left|\phi'_{\widetilde{\omega|n}} x\right|$ and bounded distortion (2.1.6) complete its proof.

Next, we will show the first assertion. Convergence of $\left|\psi_{\tau,x} y\right|$ is shown in [Zäh01, Theorem 1.(i)]. Though stated only for $x, y \in X$, the proof works on all of $\overline{V}$. It could

be modified to account for pairwise distances, but for $\left|\psi'_{\tau,x}y\right|$ we need an approach inspired by cross ratios. The equality

$$\left|\psi_{\tau,u}v\right| = \left|\phi'_\tau u\right|^{-1}\left|\phi_\tau u - \phi_\tau v\right|$$

permits us to rewrite

$$\left|\psi'_{\tau,x}y\right| = \frac{\left|\phi'_\tau y\right|}{\left|\phi'_\tau x\right|} = \frac{\left|\phi_\tau y - \phi_\tau z\right| / \left|\phi'_\tau z\right|}{\left|\phi_\tau z - \phi_\tau y\right| / \left|\phi'_\tau y\right|}\,\frac{\left|\phi'_\tau z\right|}{\left|\phi'_\tau x\right|} = \frac{\left|\psi_{\tau,z}y\right|}{\left|\psi_{\tau,y}z\right|}\,\frac{\left|\psi_{\tau,x}z\right|}{\left|\psi_{\tau,z}x\right|} \tag{4.3.2}$$

in terms of quantities which are known to converge individually at the desired rate. It remains to find uniform bounds on all the factors. The as-yet-undetermined variable z was introduced to bound the denominators away from zero: since $\overline{V}$ is connected, we can always find some $z \in \overline{V}$ such that both $|x - z|$ and $\left|y - z\right|$ exceed diam $\overline{V}/4$. Our second assertion then places bounds on the $\left|\psi_{\tau,v}u\right|$-like expressions. At the price of enlarging the constant c_ψ, this gives us both the asserted (rate of) convergence of, and bounds on $\left|\psi'_{\tau,x}y\right|$.

The next goal is convergence of $\left(\psi'_{\tau,x}y\right)^{\mathrm{orth}}$. Each of the finitely many functions $x \mapsto \phi'_i x$ is γ-Hölder continuous. Since $\inf_{i,x}\left|\phi'_i x\right| > 0$, $\left(\phi'_i x\right)^{\mathrm{orth}} = \left|\phi'_i x\right|^{-1}\phi'_i x$ also is Hölder with respect to the action invariant metric on $O(d)$. Application of the triangle inequality, action invariance, and Hölder continuity let us demonstrate the Cauchy sequence property (suppressing "orth"):

$$\begin{aligned}
& d_{O(d)}\left(\left(\phi'_{\widetilde{\omega|n+k}}x\right)^{-1}\left(\phi'_{\widetilde{\omega|n+k}}y\right), \left(\phi'_{\widetilde{\omega|n}}x\right)^{-1}\left(\phi'_{\widetilde{\omega|n}}y\right)\right) \\
\leq\; & \sum_{i=n}^{n+k-1} d_{O(d)}\left(\left(\phi'_{\widetilde{\omega|i+1}}x\right)^{-1}\left(\phi'_{\widetilde{\omega|i+1}}y\right), \left(\phi'_{\widetilde{\omega|i}}x\right)^{-1}\left(\phi'_{\widetilde{\omega|i}}y\right)\right) \\
\leq\; & \sum_{i=n}^{n+k-1} d_{O(d)}\left(\left(\phi'_{\omega_{i+1}}\phi_{\widetilde{\omega|i}}x\right)^{-1}\left(\phi'_{\omega_{i+1}}\phi_{\widetilde{\omega|i}}y\right), \mathrm{id}\right) \\
\leq\; & \sum_{i=n}^{n+k-1} \mathrm{const}\, s_{\max}^{i\gamma} \leq \mathrm{const}'\, s_{\max}^{n\gamma}.
\end{aligned}$$

By multiplying this with $\left|\psi'_{\tau,x}y\right|$, we have proved $\psi'_{\tau,x}y$ itself converges at the desired rate. Since $\psi_{\tau,x}x = 0$ and $\overline{V}$ is bounded and connected, we can represent $\psi_{\tau,x}y$ as an integral of $\psi'_{\tau,x}$ along a curve from x to y to see it converges as asserted. $\square$

Recall the shift dynamical system on $I^{\mathbb{N}} \times F \times O(d)$ is given by $\underline{\nu_{\mathrm{bi}}} = \nu_{\mathrm{bi}} \otimes \mathrm{Haar}$ and

$$\underline{\sigma_{\mathrm{bi}}}^n(\omega, x, g) = \left(\widetilde{x|n}.\omega, \sigma^n x, \left(\phi'_{x|n}\sigma^n x\right)^{\mathrm{orth}-1} g\right).$$

Let $(\eta, u) \in I^{\mathbb{N}} \times F$ be the reference point, and recall

$$G_{\eta u} = \overline{\left\langle \left(\psi'_{\widetilde{\eta},u}x\right)^{\mathrm{orth}-1}\left(\phi'_i x\right)^{\mathrm{orth}}\left(\psi'_{\widetilde{\eta},u}\phi_i x\right)^{\mathrm{orth}} : x \in F, i \in I\right\rangle_{O(d)}}.$$

Lemma 4.9. *(Ergodic fibers of group extended shift) Let $f : I^{\mathbb{N}} \times F \times O(d) \to \mathbb{R}$ be measurable and $\underline{\nu_{\mathrm{bi}}}$-integrable. The Birkhoff limit is almost everywhere given by*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f \circ \underline{\sigma_{\mathrm{bi}}}^{n} (\omega, x, g) = \int_{I^{\mathbb{N}}\times F} \int_{G_{\eta u}} f\left(\hat{v}, \hat{y}, \left(\psi'_{\tilde{\eta},u} \hat{y}\right)^{\mathrm{orth}-1} \hat{h} \left(\psi'_{\tilde{\eta},u} x\right)^{\mathrm{orth}} g\right) d\mathscr{H}_G\left(\hat{h}\right) d\nu_{\mathrm{bi}}\left(\hat{v}, \hat{y}\right), \tag{4.3.3}$$

where $\mathscr{H}_G$ is the Haar probability on $G_{\eta u}$. (It has an interpretation as the $\underline{\nu_{\mathrm{bi}}}$-conditional expectation of f on $\underline{\sigma_{\mathrm{bi}}}$-invariant sets.)

Proof. Firstly, we will summarize the theory of Brin groups according to [Dol02, Section 2.2] because that work does not assume a smooth manifold. The shift space $\left(I^{\mathbb{Z}}, \sigma_{\mathrm{bi}}, \nu_{\mathrm{bi}}\right)$ can be extended by a Hölder function $\tau : I^{\mathbb{Z}} \to H$ with values in a compact, connected Lie group H. The skew product $T(w,g) = \left(\sigma_{\mathrm{bi}} w, \tau(w) g\right)$ defines the extended dynamical system is $\left(I^{\mathbb{Z}} \times G, T, \nu_{\mathrm{bi}} \otimes \mathrm{Haar}_H\right)$. Write $\tau_n(\omega, x) := \tau\left(\sigma_{\mathrm{bi}}^{n-1} w\right) \ldots \tau(w)$ and $\tau_n^{-1}(\omega, x)$ for its point-wise inverse in H. The stable sets, unstable sets, and orbits are defined by

$$\begin{aligned}
W^s(w,g) &= \left\{\left(\overline{w}, \left[\lim_{n\to\infty} \tau_n^{-1}(\overline{w}) \tau_n(w)\right] g\right) : w_i = \overline{w}_i, i \geq N \text{ for some } N\right\},\\
W^u(w,g) &= \left\{\left(\overline{w}, \left[\lim_{n\to\infty} \tau_n \sigma_{\mathrm{bi}}^{-n}(\overline{w}) \tau_n^{-1} \sigma_{\mathrm{bi}}^{-n}(w)\right] g\right) : w_i = \overline{w}_i, i \leq -N \text{ for some } N\right\},\\
W^o(w,g) &= \left\{\left(\sigma_{\mathrm{bi}}^{n}(w), \left[\tau_n(w)\right] g\right), \left(\sigma_{\mathrm{bi}}^{-n}(w), \left[\tau_n^{-1} \sigma_{\mathrm{bi}}^{-n}(w)\right] g\right) : n \in \mathbb{N}\right\}.
\end{aligned}$$

A t-chain is a finite sequence $W = \left(w_{(1)} g_{(1)}, \ldots, w_{(n)} g_{(n)}\right)$ from $I^{\mathbb{Z}} \times H$ with $\left(w_{(i+1)}, g_{(i+1)}\right) \in W^s\left(w_{(i)}, g_{(i)}\right) \cup W^u\left(w_{(i)}, g_{(i)}\right)$, and an e-chain also allows $W^o\left(w_{(i)}, g_{(i)}\right)$. The holonomy element of W is $g(W) = g_{(n)} g_{(1)}^{-1}$; it depends only on the $I^{\mathbb{Z}}$ part in case of a t-chain. The Brin ergodicity group $\Gamma_e(w) \subseteq H$ at w is generated set-theoretically by all e-chains that start and end in w.

The system is called reduced if any two points in $I^{\mathbb{Z}}$ can be connected by a t-chain W with trivial holonomy element $g(W) = \mathrm{id}$. Then Γ_e is generated set-theoretically by all e-chains without any restrictions. Its closure $\overline{\Gamma_e} \subseteq H$ is generated by all $\tau(w)$, $w \in I^{\mathbb{Z}}$. The extended dynamical system is ergodic if and only if $H = \overline{\Gamma_e}$.

Our goal is to reduce it. An active coordinate transformation Φ of $I^{\mathbb{Z}} \times H$,

$$\Phi(w,g) = (w, \alpha(w) g) = (w', g')$$

conjugates T to T' (i.e., $\Phi \circ T = T' \circ \Phi$) if and only if it maps τ_n to

$$\tau'_n(w') = \alpha\left(\sigma_{\mathrm{bi}}^{n} w\right) \tau_n(w) \alpha^{-1}(w).$$

Thus $g'(W) = \alpha(w_{\mathrm{end}}) g(W) \alpha^{-1}\left(w_{(1)}\right)$. Denote $[u,v]$ the local product, i.e., u at negative and v at positive indices. We pick a reference point w^0, connect it to w by the t-chain $W_w = \left(w, \left[w^0, w\right], w^0\right)$, and set $\alpha(w) := g^{-1}(W_w)$, i.e.,

$$\alpha(w) = \left[\lim_{n\to\infty} \tau_n^{-1}(w) \tau_n\left(\left[w^0, w\right]\right)\right] \left[\lim_{n\to\infty} \tau_n\left(\sigma_{\mathrm{bi}}^{-n}\left[w^0, w\right]\right) \tau_n^{-1}\left(\sigma_{\mathrm{bi}}^{-n} w^0\right)\right].$$

Our coordinate transformation Φ reduces the system because $g'(W_w) = \mathrm{id}$. Therefore, the dynamical subsystem $\left(I^{\mathbb{Z}} \times \overline{\Gamma_e}, T', \nu_{\mathrm{bi}} \otimes \mathscr{H}_{\overline{\Gamma_e}}\right)$ is ergodic, where $\mathscr{H}_{\overline{\Gamma_e}}$ is the Haar probability on $\overline{\Gamma_e}$.

Secondly, we will more explicitly compute the ergodic fibres of the original dynamical system. (Our α corresponds to h of [PP97, Theorem 3] and u of [Rau07, Theorem 1.3]

and [CR09, (10)], in the sense that $\overline{\Gamma_e}\alpha$ agrees with its counterparts.) In the original coordinates, $\overline{\Gamma_e}$ and α depend on the choice of reference point:

$$\overline{\Gamma_e} = \overline{\langle \alpha(\sigma_{\mathrm{bi}}\hat{w})\,\tau(\hat{w})\,\alpha^{-1}(\hat{w}) : \hat{w} \in I^{\mathbb{Z}} \rangle_H}.$$

For all $h \in H$, define the function

$$f_h(w,g) := f(w,gh)$$

on $I^{\mathbb{Z}} \times H$. Birkhoff's ergodic theorem for the ergodic, reduced subsystem $I^{\mathbb{Z}} \times \overline{\Gamma_e}$ implies that $f_h \circ \Phi^{-1}$ converges on $\mathscr{H}_{\overline{\Gamma_e}}$-almost all $\alpha(w)\,g \in \overline{\Gamma_e}$:

$$\frac{1}{n}\sum_{l<n} f \circ T^l(w,gh) = \frac{1}{n}\sum_{l<n}\left[f_h \circ \Phi^{-1}\right] \circ (T')^l \circ \Phi(w,g) \underset{n\to\infty}{\longrightarrow}$$

$$\int_{I^{\mathbb{Z}}}\int_{\overline{\Gamma_e}} f_h \circ \Phi^{-1}(\hat{w},\hat{g})\,d\mathscr{H}_{\overline{\Gamma_e}}(\hat{g})\,d\nu_{\mathrm{bi}}(\hat{w}) = \int_{I^{\mathbb{Z}}}\int_{\overline{\Gamma_e}} f\left(\hat{w},\alpha(\hat{w})^{-1}\,\hat{g}\,\alpha(w)\,gh\right)d\mathscr{H}_{\overline{\Gamma_e}}(\hat{g})\,d\nu_{\mathrm{bi}}(\hat{w}).$$

We have used $T = \Phi^{-1} \circ \Phi \circ T = \Phi^{-1} \circ T' \circ \Phi$ to insert Φ into the first line. In the last equality, we have replaced h with $\alpha(w)\,gh$ because $\mathscr{H}_{\overline{\Gamma_e}}$ is action invariant.

Birkhoff's theorem can also be applied to f on the original dynamical system $I^{\mathbb{Z}} \times H$. The above ergodic sum converges to the ergodic fibre average for $\mathscr{H}_H$-almost every gh. We still need to show $\mathscr{H}_H$-almost every $gh \in H$ can be written using an h-dependent, $\mathscr{H}_{\overline{\Gamma_e}}$-full set of $\alpha(w)\,g \in \overline{\Gamma_e}$. But Federer's coarea formula [KP08, Theorem 5.4.9] proves $\mathscr{H}_H$ can be obtained by integrating right translates of $\alpha(w)^{-1}\,\mathscr{H}_{\overline{\Gamma_e}}$ because any Haar probability is a Hausdorff measure.

Thirdly, we will apply the above theory. The positive (right) index part of $(\omega,x) \in I^{\mathbb{Z}}$ will be given by $x \in F$ and the negative (left) one by $\omega \in I^{\mathbb{N}}$. Both the thermodynamic potential $\ln\left|\phi'_{x|1}\sigma x\right|$ and $\left(\phi'_{x|1}\sigma x\right)^{\mathrm{orth}}$ are Lipschitz continuous on $H := O(d)$ with respect to

$$d_\theta\left((\omega,x),(\eta,u)\right) = \max\{\theta^n : \omega_n \neq \eta_n \text{ or } x_n \neq u_n\}$$

if $\theta := s_{\max}^{\gamma}$, as required. We set

$$\tau_n(\omega,x) := \left(\phi'_{x|n}\sigma^n x\right)^{\mathrm{orth}-1} = \left(\phi'_{x_n}\sigma^n x\right)^{\mathrm{orth}-1} \dots \left(\phi'_{x_1}\sigma x\right)^{\mathrm{orth}-1},$$

and $\tau_n\sigma_{\mathrm{bi}}^{-n}(\omega,x) = \left(\phi'_{\widetilde{\omega|n}}x\right)^{\mathrm{orth}-1}$. Let (η,u) be our reference point w^0. The t-chain $\left(w,[w^0,w],w^0\right)$ becomes $\left((\omega,x),(\eta,x),(\eta,u)\right)$. Note τ_n depends only on x, so the stable set is flat, i.e., $(\eta,x,g) \in W^s(\omega,x,g)$. We compute

$$\alpha(\omega,x) = \left[\lim_{n\to\infty}\mathrm{id}\right]\left[\lim_{n\to\infty}\left(\phi'_{\widetilde{\eta|n}}x\right)^{\mathrm{orth}-1}\left(\phi'_{\widetilde{\eta|n}}u\right)^{\mathrm{orth}}\right] = \left(\psi'_{\tilde{\eta},u}x\right)^{\mathrm{orth}}$$

(by Proposition 4.8) and use the formulas. □

4.4. **Renewal radii and covariant neighborhood net.** The next two sections have more geometrical content. They generalize the methods from the self-similar case ([BZ11, RZ10]) and unify several notions of curvature densities (Remark 4.12).

We will later split the dr-integral in (4.1.1) into chunks of comparable level of detail of F "visible" through the parallel set F_r. More precisely, near a point $x \in F$, the parallel set width $R(x,m)$ is "well-adapted" to magnification under $\phi_{x|m}^{-1}$. Intuitively, $R(x,m) \sim \operatorname{dist}\left(x,\phi_{x|m}X^c\right)$. "Well-adapted" means we can localize the curvature measures because no other level set $\left(\phi_\tau F\right)_{R(x,m)}$, $\tau \in I^m$, overlaps $\left(\phi_{x|m}F\right)_{R(x,m)}$ in an $R(x,m)$-ball around x. This requirement and covariance under $\phi_{x|m}$ naturally lead to our definition of the

renewal radius $R(x,m)$. For technical reasons, we will integrate a tent function A around x instead the curvature of the $R(x,m)$-ball itself. The support of $z \mapsto A(|x-z|, R(x,m))$ is contained in a slightly larger ball because A needs to sample an environment of $\partial(F_r)$.

Recall $1 < a$ and $0 < \epsilon_{\max} \le a^{-1} \operatorname{dist}(F, V^c)$.

Definition 4.10. Define for μ-almost all $x \in F$ and $m \in \mathbb{N}$: $R(x,-1) := \epsilon_{\max}$,

$$\begin{aligned} \widetilde{R}(x,0) &:= \frac{\operatorname{dist}(x, X^c)}{2K_\phi a} \min\left\{1, \frac{2K_\phi^{-1} a \epsilon_{\max}}{\max_{x \in F} \operatorname{dist}(x, X^c)}\right\}, \\ R(x,m) &:= \sup_{n \ge m} \left|\phi'_{x|n} \sigma^n x\right| \widetilde{R}(\sigma^n x, 0). \end{aligned}$$

We could eliminate the minimum in the definition of $\widetilde{R}(x,0)$ if we replaced the (OSC) set X with its iterated image under the IFS.

Lemma 4.11. *(Renewal radii) For μ-almost all $x, y \in F$, all $m \ge 0$, $z \in V$:*

(1) *Avoids overlap sets $(\phi_\tau F)_r \cap (\phi_\vartheta F)_r$ $(\tau, \vartheta \in I^m)$:*

$$\begin{aligned} |z-x| < aR(x,m) &\implies z \notin (\phi_{x|m} X^c)_{R(x,m)}, \\ \left|y-x\right| < 2aR(x,m) &\implies x|m = y|m, \end{aligned}$$

(2) *Covariance:* $R(x,m) = \left|\phi'_{x|m} \sigma^m x\right| R(\sigma^m x, 0)$,

(3) *Monotonicity:* $a^{-1} \operatorname{dist}(F, V^c) \ge \epsilon_{\max} \ge R(x,m) \ge R(x,m+1) > 0$,

(4) *Limit:* $R(x,m) \underset{m\to\infty}{\longrightarrow} 0$,

(5) *Integrability:* $\int_F \left|\ln R(y,0)\right| d\mu(y) < \infty$ *and, for some $c > 1$:*

$$\int_F c^{\left|\ln R(y,0)\right|} d\mu(y) < \infty. \tag{4.4.1}$$

Proof. (Compare [Pat04].) The first assertion holds if any $y \in V$ avoids $\phi_{x|m} X^c$ if $\left|y-x\right| < 2aR(x,m)$. Indeed: given z, set $y := 2z - x$. Given $y \in F$, (OSC) implies $y \in \phi_{x|m} F$.

We use the Open Set Condition, the triangle inequality, bounded distortion (2.1.6) and an undetermined $0 < c_R \le 1$:

$$\begin{aligned} \operatorname{dist}(y, \phi_{x|m} X^c) &\overset{\text{(OSC)}}{=} \sup_{n \ge m} \operatorname{dist}(y, \phi_{x|n} X^c) \ge \sup_{n \ge m} \operatorname{dist}(x, \phi_{x|n} X^c) - \left|x-y\right| \\ &> \sup_{n \ge m} \operatorname{dist}(\phi_{x|n} \sigma^n x, \phi_{x|n} X^c) - 2aR(x,m) \\ &\overset{(2.1.6)}{\ge} K_\phi^{-1} \sup_{n \ge m} \left|\phi'_{x|n} \sigma^n x\right| \operatorname{dist}(\sigma^n x, X^c) - \frac{2a}{c_R} R(x,m) := 0. \end{aligned}$$

We solve the last equation for $R(x,m)$,

$$R(x,m) = \frac{c_R}{2aK_\phi} \sup_{n \ge m} \left|\phi'_{x|n} \sigma^n x\right| \operatorname{dist}(\sigma^n x, X^c),$$

and recuperate the original definition of $R(x,m)$ if $c_R \le 1$ is the largest value such that $\max_{x \in F} R(x,0) \le \epsilon_{\max}$. (As an aside, $\widetilde{R}(x,0) \le R(x,0) \le K_\phi^2 \widetilde{R}(x,0)$.)

The second assertion (covariance) follows from, $(n \ge m)$

$$\left|\phi'_{x|n} \sigma^n x\right| = \left|\phi'_{x|m} \sigma^m x\right| \left|\phi'_{(\sigma^m x)|n-m} \sigma^{n-m} (\sigma^m x)\right|,$$

and the definitions of $R(x,m)$ and $R(\sigma^m x,0)$. Monotonicity is trivial except for $R(x,m)>0$, which follows from the integrability. The limit assertion stems from

$$R(x,m)=\left|\phi'_{x|m}\sigma^m x\right| R(\sigma^m x,0)\le s_{\max}^m\epsilon_{\max}.$$

Integrability of $c^{\left|\ln R(y,0)\right|}$ is reduced by $c'\operatorname{dist}\left(y,X^c\right)=\widetilde{R}\left(y,0\right)\le R\left(y,0\right)\le\epsilon_{\max}$ to

$$\int c^{\left|\ln\operatorname{dist}\left(y,X^c\right)\right|}d\mu\left(y\right)<\infty. \tag{4.4.2}$$

[Pat04, Lemma 3.8] proves $\int\left|\ln\operatorname{dist}(\cdot,X^c)\right|d\mu$ is finite using (SOSC). Our case is similar. Patzschke dominates the integral by a geometric series. We can safely increase its base while (in Patzschke's notation) $0<\ln c<\left|\ln\left(C_2^{-1}\left\|S'_\eta\right\|r_{\min}\right)\right|\left|\ln\left(1-C_5^{-1}\psi\left[\eta\right]\right)\right|$. $\square$

The purpose of A is to sample the curvature located in a small ball $B(x,ar)$, $r<R(x,m)$. It has to work together with weak convergence of curvature measures (Section 4.6). So we define it as a continuous counterpart of $1_{B(x,ar)}(z)$: the tent function

$$A(|x-z|,r):=\max\left(1-\frac{|x-z|}{ra},0\right).$$

Many other choices are viable due to (4.1.2). So we will treat A as a black box and demand only the next lemma holds.

Remark 4.12. Our A unifies previous approaches in the self-similar setting. It can be replaced with any of the locally homogeneous neighborhood nets $A_F^{[\mathrm{RZ10}]}(x,r)$ from [RZ10, Example 2.1.1] if $A(|x-z|,r):=1_{A_F^{[\mathrm{RZ10}]}(x,r)}(z)$. (Our continuity is replaced with their Lemma 2.1.3.) This includes a net that eliminates the denominator A in ρ and π.

Items 1, 3 are designed to enable Fubini (Proposition 4.2); 2, 4, 5 to localize and preimage curvature (Lemma 4.14); and 3, 6 to preserve weak convergence (Lemma 4.16).

Lemma 4.13. *(Covariant neighborhood net) For almost all $x,\hat{x},y,\hat{y}\in F$, all $u,\hat{z},z\in V$, and almost all $r,\hat{r}\le\epsilon_{\max}$:*

(1) *(Measurability) $(x,z,r)\mapsto A(|x-z|,r)$ is Borel measurable;* $0\le A(|x-z|,r)\le 1$.
(2) *(Locality)*

$$\left\{\hat{z}\in\mathbb{R}^d:A(|\hat{x}-\hat{z}|,\hat{r})\neq 0\right\}\subseteq\mathring{B}(\hat{x},a\hat{r})\subseteq V$$

.

(3) *(D-set property) There is a constant C_A, $0<C_A<\infty$, such that*

$$C_A^{-1}\le\frac{\int A\left(\left|\hat{y}-\hat{z}\right|,\hat{r}\right)d\mu\left(\hat{y}\right)}{\hat{r}^D}\le C_A.$$

More generally, let write $\tau\in I^*$ and $\psi_{\tau,u}z\in\left(\left(\psi_{\tau,u}F\right)_r\right)$. Then

$$C_A^{-1}\le\frac{\int A\left(\left|\psi_{\tau,u}y-\psi_{\tau,u}z\right|,r\right)d\mu\left(y\right)}{r^D}\le C_A. \tag{4.4.3}$$

(4) *(2-chain of supports avoids intersections $\left[\phi_\tau F\right]_{\hat{r}}\cap\left[\phi_{\hat{x}|m}F\right]_{\hat{r}}$) If $\hat{r}\le R(\hat{x},m)$,*

$$A(|\hat{x}-\hat{z}|,\hat{r})\neq 0 \text{ and } A\left(\left|\hat{y}-\hat{z}\right|,\hat{r}\right)\neq 0\Longrightarrow\hat{x}|m=\hat{y}|m.$$

(5) *(Covariance w.r.t the ψ-distorted distance; compatibility with the dynamics) If $\hat{r} \le R(\hat{x}, m)$ and $\phi_{\hat{x}|m} z \in \left(\left(\phi_{\hat{x}|m} F\right)_{\hat{r}}\right)$, then*

$$A\left(\left|\hat{y} - \phi_{\hat{x}|m} z\right|, \hat{r}\right) = A\left(\left|\psi_{\hat{x}|m,u} \sigma^m y - \psi_{\hat{x}|m,u} z\right|, \frac{\hat{r}}{\left|\phi'_{\hat{x}|m} u\right|}\right) 1\{\hat{y}|m = \hat{x}|m\}.$$

(6) *(Lipschitz continuity) As a function $|x-z| \mapsto A(|x-z|, r)$ for fixed r,*

$$\mathrm{Lip}(A) \le r^{-1}.$$

Proof. Point one and six are obvious. The second and fourth follow from Lemma 4.11 point (1).

The third assertion will be proved in the version with $\psi_{\tau,u}$. It contains the simpler version for $\tau = \emptyset$, $\psi_{\tau,u} = \mathrm{id}$. Abbreviate $\psi := \psi_{\tau,u}$. We want to estimate

$$\begin{aligned}
\int_F A\left(\left|\psi y - \psi z\right|, r\right) d\mu(y) &= \int_0^1 \int_F 1\left\{1 - t \ge A\left(\left|\psi y - \psi z\right|, r\right)\right\} d\mu(y)\, dt \\
&= \int_0^1 \int_F 1\left\{tar \ge \left|\psi y - \psi z\right|\right\} d\mu(y)\, dt.
\end{aligned}$$

There is a point $x_z \in F$ such that $\left|\psi z - \psi x_z\right| \le r$. The triangle inequality implies

$$\begin{array}{ccc}
\left|\psi y - \psi z\right| - r \le & \left|\psi y - \psi x_z\right| \le & \left|\psi y - \psi z\right| + r, \\
\left\{tar + r \ge \left|\psi y - \psi x_z\right|\right\} \ge & 1\left\{tar \ge \left|\psi y - \psi z\right|\right\} \ge & 1\left\{tar - r \ge \left|\psi y - \psi x_z\right|\right\},
\end{array}$$

and then Proposition 4.8 (2)

$$1\left\{(ta+1) r K_\psi \ge \left|y - x_z\right|\right\} \ge \quad 1\left\{tar \ge \left|\psi y - \psi z\right|\right\} \ge \quad 1\left\{(ta-1) r / K_\psi \ge \left|y - x_z\right|\right\}.$$

We integrate w.r.t. $d\mu(y)\, dt$ and then use the D-set property (2.1.4) of μ for balls centered on x_z,

$$\int_0^1 C_F \left((ta+1) r K_\psi\right)^D dt \ge \quad \int_F A\left(\left|\psi y - \psi z\right|, r\right) d\mu(y) \ge \quad \int_{1/a}^1 C_F^{-1} \left((ta-1) r / K_\psi\right)^D dt.$$

This inequality fits between $C_A r^D$ and $C_A^{-1} r^D$ for a suitable C_A.

To prove the fifth assertion, we will use $\left[\phi'_{\hat{x}|m} u\right]^{-1} \circ \phi_{\hat{x}|m} = \psi_{\hat{x}|m,u}$ up to an additive constant (Proposition 4.8 (3)):

(4.4.4)

$$\left|\phi'_{\hat{x}|m} u\right|^{-1} \left|y - \phi_{\hat{x}|m} z\right| = \left|\left[\phi'_{\hat{x}|m} u\right]^{-1} \left(\phi_{\hat{x}|m} \sigma^m \hat{y} - \phi_{\hat{x}|m} z\right)\right| = \left|\psi_{\hat{x}|m,u} \circ \sigma^m \hat{y} - \psi_{\hat{x}|m,u} z\right|.$$

Inserting this into (2.3.1) results in the assertion. □

4.5. **Localizing and scaling the curvature under the inverse IFS.** As $F_r = \bigcup_{\tau \in I^m} \left(\phi_\tau F\right)_r$, parts of F_r agree with $\left(\phi_\tau F\right)_r$. Under the similarity $\phi'_\tau u$, this is the image of another parallel set of $\psi_{\tau,u} F$. For given $m \in \mathbb{N}$ and in a small area around $x \in F$, the agreeing r are less than $R(x, m)$. This idea also works for the associated curvature measure instead of the set. Note ρ, π below contain the chunk of curvature between $r \le R(x, m)$ and $r \not\le R(x, m+1)$.

(The curvature of $(\psi F)_r$ is a combination of those of $\psi^{-1}\left((\psi F)_r\right)$, compare [Ber03b]. But this does not help us.)

Note

$$\rho^{k,\pm}_{(R(x,m+1),R(x,m)]}(x,r,f) = \int q_m(\mathrm{id},x,z,r)\, f(\underline{z})\; dC^{\pm}_k(F_r,\underline{z}), \tag{4.5.1}$$

$$\pi^{k,\pm}_{\tau}(x,r,f) = \int q_0(\psi_{\tau,u},x,z,r)\, f(\underline{z})\; dC^{\pm}_k\left((\psi_{\tau,u}F)_r,\psi_{\tau,u}\underline{z}\right), \tag{4.5.2}$$

and with a placeholder ψ,

$$q_m(\psi,x,z,r) := 1_{(R(x,m+1),R(x,m)]}\left(\frac{r}{|\psi' x|}\right)\frac{r^{D-k}A\left(\left|\psi x-\psi z\right|,r\right)}{\int |\psi' y|^D A\left(\left|\psi y-\psi z\right|,r\right)\, d\mu(y)}. \tag{4.5.3}$$

Lemma 4.14. *Given $\hat{x}\in F$, $u\in V$, $m\in\mathbb{N}$, the substitutions*

$$x := \sigma^m\hat{x},\quad \tau := \hat{x}|m,\quad r := \hat{r}/\left|\phi'_\tau u\right|$$

transform π and ρ into each other:

$$\pi^{k,\pm}_{\tau}(x,r,f\circ\phi_\tau) = \rho^{k,\pm}_{(R(\hat{x},m+1),R(\hat{x},m)]}(\hat{x},\hat{r},f). \tag{4.5.4}$$

Proof. Abbreviate $\rho := \rho^{k,\pm}_{(R(\hat{x},m+1),R(\hat{x},m)]}(\hat{x},\hat{r},f)$. For clarity, note that

$$\hat{x} = \phi_\tau x,\ m = |\tau|,\ \hat{r} = r\left|\phi'_{\hat{x}|m}u\right|.$$

In the first half of the proof, we will rewrite ρ's curvature density $q_m(\mathrm{id},\hat{x},\hat{z},\hat{r})$ in terms of the new variables

$$y := \phi_\tau^{-1}\hat{y},\quad r = \hat{r}/\left|\phi'_\tau u\right|,\quad \underline{z} = \phi_\tau^{-1}\underline{\hat{z}}.$$

They will later leave the measure $\hat{r}^{-1}d\hat{r}$ unchanged. This means $\hat{x}$, $\hat{y}$, $\underline{\hat{z}}$, and $\hat{r}$ are preimaged m times under the IFS.

$R(\hat{x},m)$ was defined to be covariant:

$$1\{R(\hat{x},m+1) < \hat{r} \le R(\hat{x},m)\} = 1\left\{\left|\phi'_\tau\sigma^m\hat{x}\right| R(\sigma^m\hat{x},1) < r\left|\phi'_\tau u\right| \le \left|\phi'_\tau\sigma^m\hat{x}\right| R(\sigma^m\hat{x},0)\right\}$$

$$= 1\left\{R(x,1) < r\frac{\left|\phi'_\tau u\right|}{\left|\phi'_\tau x\right|} \le R(x,0)\right\} = 1\left\{R(x,1) < \frac{r}{\left|\psi'_{\tau,u}x\right|} \le R(x,0)\right\} \tag{4.5.5}$$

In the denominator of $q_m(\mathrm{id},\hat{x},\hat{z},\hat{r})$, Lemma 4.13 (4) tells us

$$\hat{y}|m = \hat{x}|m.$$

Therefore, we can transform both instances of A with the same τ, see (4.4.4) in the same lemma:

$$\begin{aligned} A(|\hat{x}-\hat{z}|,\hat{r}) = A\left(\left|\hat{x}-\phi_\tau z\right|,\hat{r}\right) &= A\left(\left|\psi_{\tau,u}x-\psi_{\tau,u}z\right|,r\right)1\{\tau=\hat{x}|m\},\\ A\left(\left|\hat{y}-\hat{z}\right|,\hat{r}\right) = A\left(\left|\hat{y}-\phi_\tau z\right|,\hat{r}\right) &= A\left(\left|\psi_{\tau,u}y-\psi_{\tau,u}z\right|,r\right)1\{\tau=\hat{y}|m\}.\end{aligned} \tag{4.5.6}$$

(A geometric interpretation: the covariance of A was designed to match that of the curvature.) We integrate the last two lines. Then the Perron-Frobenius operator of μ introduces $d\mu\circ\sigma^{-m}/d\mu=\left|\phi_\tau' y\right|^D$. That is absorbed into $\psi_{\tau,u}$,

$$\begin{aligned}
\int A\left(\left|\hat y-\hat z\right|,\hat r\right)\,d\mu\left(\hat y\right) &= \int A\left(\left|\psi_{\tau,u}\sigma^m\hat y-\psi_{\tau,u}z\right|,r\right)1_{\{\tau=\hat y|m\}}\,d\mu\left(\hat y\right)\\
&= \int \left|\phi_\tau' y\right|^D A\left(\left|\psi_{\tau,u}y-\psi_{\tau,u}z\right|,r\right)\,d\mu\left(y\right)\\
&= \int \left|\psi_{\tau,u}' y\right|^D A\left(\left|\psi_{\tau,u}y-\psi_{\tau,u}z\right|,r\right)\,d\mu\left(y\right)\left|\phi_\tau' u\right|^D. \qquad (4.5.7)
\end{aligned}$$

The remaining factor of $q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)$ transforms as

$$\hat r^{D-k}=r^{D-k}\left|\phi_\tau' u\right|^D\left|\phi_\tau' u\right|^{-k}.$$

We assemble both factors above and (4.5.5), (4.5.6), (4.5.7) to obtain

$$\left|\phi_\tau' u\right|^k q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)=q_0\left(\psi_{\tau,u},x,z,r\right). \qquad (4.5.8)$$

In the second half of the proof, we will make use of the locality (2.2.3) and scaling (2.2.2) properties of curvature measures to perform the remaining substitution $\underline{\hat z}\mapsto\psi_{\tau,u}\underline{z}$. First, we rename $\underline{\hat z}$ to $\phi_\tau\underline{z}$,

$$\rho=\int q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)f\left(\phi_\tau\underline{z}\right)\,dC_k^{\pm}\left(F_{\hat r},\phi_\tau\underline{z}\right),$$

To localize, only a small area around $\hat x$ supports $\hat z\mapsto q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)$. This is because A lives on $\mathring{B}(\hat x,aR(\hat x,m))$ (Lemma 4.13 (2)), which is disjoint from $\left(\phi_{v|m}F\right)_{\hat r}$ whenever $\tau:=\hat x|m\neq v|m\in I^m$ (Lemma 4.11 (1)). Therefore, we may replace $F_{\hat r}$ with $\left(\phi_\tau F\right)_{\hat r}$ inside the curvature measure:

$$\rho=\int q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)f\left(\phi_\tau\underline{z}\right)\,dC_k^{\pm}\left(\left(\phi_\tau F\right)_{\hat r},\phi_\tau\underline{z}\right).$$

Since $\psi_{\tau,u}\circ\phi_\tau^{-1}$ is a similarity with Lipschitz constant $\left|\phi_\tau' u\right|^{-1}$, it maps parallel sets to parallel sets,

$$\psi_{\tau,u}\circ\phi_\tau^{-1}\left(\left(\phi_\tau F\right)_{\hat r}\right)=\left(\psi_{\tau,u}F\right)_{\hat r/\left|\phi_\tau' u\right|}=\left(\psi_{\tau,u}F\right)_r.$$

We may introduce the similarity via the scaling property (2.2.2) of curvature measures,

$$\begin{aligned}
\rho &= \left|\phi_\tau' u\right|^k\int q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)f\left(\phi_\tau\underline{z}\right)\,dC_k^{\pm}\left(\psi_{\tau,u}\circ\phi_\tau^{-1}\left(\left(\phi_\tau F\right)_{\hat r}\right),\psi_{\tau,u}\circ\phi_\tau^{-1}\left(\phi_\tau\underline{z}\right)\right)\\
&= \left|\phi_\tau' u\right|^k\int q_m\left(\mathrm{id},\hat x,\hat z,\hat r\right)f\circ\phi_\tau\left(\underline{z}\right)\,dC_k^{\pm}\left(\left(\psi_{\tau,u}F\right)_r,\psi_{\tau,u}\underline{z}\right).
\end{aligned}$$

(Scaling also proves $C_k^{\pm}\left(\left(\psi_{\tau,u}F\right)_r,\cdot\right)$ is measurable as a function of r on the set where we need it, see Remark 4.1) This combines with (4.5.8) to complete the proof. □

4.6. **Curvature of distorted images converges.** In the last section, we compared F_r to a (locally) fixed range of parallel sets of the distorted fractal $\psi_{\widetilde{\omega|n},u}F$. Unlike the self-similar case, we cannot eliminate the distortion $\psi_{\widetilde{\omega|n},u}$ altogether. But this map converges as $n\to\infty$ and depends asymptotically only on the "dynamical" variable $(\omega,u)\in I^{\mathbb N}\times F$. In this section, we prove the associated "chunk of curvature" $\pi_{x|n}^{(k,\pm)}$ converges, too.

The next continuity result will be applied to $K^n := \psi_{\widetilde{\omega|n},u} F$ and $K^\infty := \psi_{\tilde{\omega},u} F$ for $u \in F$, $\omega \in I^{\mathbb{N}}$. If we had a corresponding result about the variations $C_k^{\pm}$, our main theorem would hold for those, too (Remark 4.4). It seems difficult to quantify the rate of convergence unless the reach can be controlled, compare [CSM06, Corollary 2, p. 384] in the smooth setting.

Fact 4.15. *Let K^∞, $K^n \subseteq \mathbb{R}^d$ be a sequence of nonempty, compact sets, and suppose K^n converges to K^∞ in the Hausdorff metric, i.e.,*

$$\lim_{n\to\infty} \inf\left\{ r > 0 : K^n_r \supseteq K^\infty \text{ and } K^n \subseteq K^\infty_r \right\} = 0.$$

(1) *Let $k \in \{d-1, d\}$. For all $r > 0$ except possibly a countable set,*

$$\operatorname{wlim}_{n\to\infty} C_k\left(\widetilde{K^n_r}, \cdot \times S^{d-1}\right) = C_k\left(\widetilde{K^\infty_r}, \cdot \times S^{d-1}\right).$$

(2) *Let $k \in \{0, \ldots, d-1\}$, and assume $\widetilde{K^\infty_r}$ has positive reach for a fixed $r > 0$. Then $\widetilde{K^n_r}$ has uniformly positive reach for any sufficiently large n, and*

$$\operatorname{wlim}_{n\to\infty} C_k\left(\widetilde{K^n_r}, \cdot\right) = C_k\left(\widetilde{K^\infty_r}, \cdot\right).$$

Proof. The first assertion is due to Stacho ([Sta76, Theorem 3]). It was put in the present form by [RSM09, Theorem 5.1], [RW10, Corollary 2.5]. The second assertion can be found in [RSM09, Theorem 5.2] for (non-directional) Federer curvatures only. The same work shows $\liminf_{n\to\infty} \operatorname{reach} \widetilde{K^n_r} > 0$. This is enough for the normal current of $\widetilde{K^n_r}$ to converge in the flat semi-norms ([RZ01, Theorem 3.1]). Since Lipschitz-Killing curvature has a current representation ([Zäh86, (18)]), it converges weakly. □

Recall q (4.5.3) is a template for the integrands of $\pi_\tau^{k,\pm}(x, r, f)$ (4.5.2) for finite $\tau \in I^*$ and $\pi_{\tilde{\omega}}^k(x, r, f)$ (2.3.6) for infinite $\omega \in I^{\mathbb{N}}$,

$$q_m(\psi, x, z, r) = 1_{(R(x,m+1),R(x,m)]}\left(\frac{r}{|\psi' x|}\right) \frac{r^{D-k} A\left(\left|\psi x - \psi z\right|, r\right)}{\int \left|\psi' y\right|^D A\left(\left|\psi y - \psi z\right|, r\right) d\mu(y)}.$$

Lemma 4.16. *Let $(\omega, x) \in I^{\mathbb{N}} \times F$, $\underline{z} \in V \times S^{d-1}$, fix a reference point $u \in V$, and assume $r > 0$ is regular in the sense of Assumption 3.1.*

(1) *As a weak limit of signed measures,*

$$\pi_{\tilde{\omega}}^k(x, r, \cdot) = \operatorname{wlim}_{n\to\infty} \pi_{\widetilde{\omega|n}}^k(x, r, \cdot). \tag{4.6.1}$$

(2) *The limit*

$$q_0\left(\psi_{\tilde{\omega},u}, x, z, r\right) = \lim_{n\to\infty} q_0\left(\psi_{\widetilde{\omega|n},u}, x, z, r\right)$$

converges uniformly in z (possibly except at two values of r). There is even a constant $c_q > 0$ such that it converges at the universal rate

$$c_q\left(1 + \operatorname{dist}\left(r\left|\psi'_{\tilde{\omega},u} x\right|^{-1}, \{R(x,1), R(x,0)\}\right)^{-1}\right) r^{-k-1} s_{\max}^{n\gamma}.$$

Proof. First, we will prove q converges at the desired rate. It is enough to know every factor and its denominator are bounded and converge at such a rate (Fact 4.7). Both $\psi'_{\widetilde{\omega|n},u} x$ and $\psi_{\widetilde{\omega|n},u} x$ converge at the rate $c_\psi s_{\max}^{n\gamma}$ (Proposition 4.8 item (1)). The factor

$$1_{(R(x,1),R(x,0)]}\left(r\left|\psi'_{\widetilde{\omega|n},u} x\right|^{-1}\right) r^{-k}$$

is bounded by r^{-k}. Only at the endpoints of the interval can it be discontinuous. To see how far its argument can stray,

$$\left| r\left|\psi'_{\widetilde{\omega|n},u}x\right|^{-1} - r\left|\psi'_{\widetilde{\omega},u}x\right|^{-1}\right| = r\left|\psi'_{\widetilde{\omega},u}x\right|^{-1}\left|\psi'_{\widetilde{\omega|n},u}x\right|^{-1}\left|\left|\psi'_{\widetilde{\omega},u}x\right| - \left|\psi'_{\widetilde{\omega|n},u}x\right|\right| \le r\frac{c_\psi}{K^2}s_{\max}^{n\gamma}.$$

The bound r^{-k} originating from the supremum norm at the worst n determines its overall rate $\frac{c_\psi r}{K^2\delta r^k}s_{\max}^{n\gamma}$, where δ is the distance from $r\left|\psi'_{\widetilde{\omega|n},u}x\right|^{-1}$ to the nearest endpoint (which can be estimated by that of $r\left|\psi'_{\widetilde{\omega},u}x\right|^{-1}$).

The next factor $A\left(\left|\psi_{\widetilde{\omega|n},u}x - \psi_{\widetilde{\omega|n},u}z\right|, r\right)$ is trivially bounded by 1, and converges at a rate $r^{-1}c_\psi s_{\max}^{n\gamma}$ due to $\mathrm{Lip}(A) \le r^{-1}$ (Lemma 4.13 (6)). The estimate (Lemma 4.13 (3))

$$c_\psi^{-1}C_A^{-1} \le \frac{\int\left|\psi'\right|^D A\left(\left|\psi_{\widetilde{\omega|n},u}x - \psi_{\widetilde{\omega|n},u}z\right|, r\right)d\mu(y)}{r^D} \le c_\psi C_A$$

makes sure we can divide without losing rate of convergence. Straightforward reasoning treats the last formula's numerator. Convergence of $q_0\left(\psi_{\widetilde{\omega|n},u}, x, z, r\right)$ is proved.

In preparation for the first item, let $\mu_n \to \mu_\infty$ be any sequence of weakly converging measures, and $g_n \to g_\infty$ norm converging, continuous functions. Note $\sup_m\left\|\mu_m\right\| < \infty$ due to Banach-Steinhaus. Then

$$\left|\mu_n(g_n) - \mu_\infty(g_\infty)\right| \le \left|(\mu_n - \mu_\infty)(g_\infty)\right| + \sup_m\left\|\mu_m\right\|\left\|g_n - g_\infty\right\|_\infty \tag{4.6.2}$$

permits us to treat the functions and measures separately. Back to $\mu_n := \pi^k_{\widetilde{\omega|n}}(x, r, f)$, we have proved its integrand $g_n(\underline{z}) := q_0\left(\psi_{\widetilde{\omega|n},u}, x, z, r\right)f(\underline{z})$ converges uniformly. Positive reach of $\left(\psi_{\widetilde{\omega},u}F\right)_r$ (Assumption 3.1 (3)) and Fact 4.15 (and equicontinuously uniform convergence of $\psi_{\widetilde{\omega|n},u}$ inside $d\underline{z}$) confirm the curvature measure $\mu_n(d\underline{z}) := C_k\left(\left(\psi_{\widetilde{\omega|n},u}F\right)_r, d\psi_{\widetilde{\omega|n},u}\underline{z}\right)$ converges. (The point-wise limit $\pi_{\widetilde{\omega}}(x, r, f)$ remains measurable as a function of ω, x, r, see Remark 4.1). □

4.7. **Sufficient conditions for the uniform integrability assumption.** Our main result depends on a uniform integrability assumption. We can always prove it for C_d (Minkowski content) and C_{d-1} (surface content) since these contain no principal curvatures. More generally, Lemma 4.17 below carefully estimates a dominating function. Its resulting order of integrability is necessary in the self-similar case. The converse dominated ergodic theorem [Pet89, Theorem 3.1.16] shows this when applied to [BZ11, (3.4.1)] and the suspension flow.

Uniform integrability singles out μ_{bi} and ν_{bi} as the geometrically "correct" limit measures. Recall their thermodynamic potential is the contraction ratio $x \mapsto -D\log\phi_{x_1}\sigma x$. Measures generated by arbitrary potentials are studied in multifractal formalism [Pat97]. At the price of an extra multiplier, these could replace μ_{bi} and ν_{bi} up to here. But different potentials make mutually singular measures. So fractal curvature can have a density with respect to at most one of them.

Lemma 4.17. *The uniform integrability assumption* (3.0.1) *of the theorem holds if the function*
(4.7.1)

$$h(r,\omega,x)=r^{-k}\sup_{M>0}\frac{1}{M}\sum_{m=0}^{M-1}C_k^{\mathrm{var}}\left(\left(\psi_{\widetilde{\omega|m},u}F\right)_r,\psi_{\widetilde{\omega|m},u}\mathring{B}(x,ar)\right)1_{(R(x,1),R(x,0)]}\left(\frac{r}{\left|\psi'_{\widetilde{\omega|m},u}x\right|}\right)$$

is in the Zygmund space, i.e.,

$$\int_{I^{\mathbb{N}}\times F}\int_0^\infty\max\{0,|h(r,\omega,x)|\ln|h(r,\omega,x)|\}\,\frac{dr}{r}\,d\nu_{\mathrm{bi}}(\omega,x)<\infty$$

and if also

$$\int_{I^{\mathbb{N}}\times F}\int_0^\infty\sup_{0<\epsilon\le\frac{\epsilon_{\max}}{2}}\frac{1_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(r)\,C_k^{\mathrm{var}}\left(F_r,\mathring{B}(x,ar)\right)}{\ln\frac{\epsilon_{\max}}{\epsilon}\,r^k}\,\frac{dr}{r}\,d\nu_{\mathrm{bi}}(\omega,x)<\infty.$$

Note $\left|\psi'_{\widetilde{\omega|m},x}x\right|=1$ simplifies h in case u is replaced with x, see Theorem 3.3.

Proof. We prepare to integrate a dominating function, i.e., the supremum of (3.0.1) over all $\epsilon\in(0,\epsilon_{\max}/2]$. First, we eliminate ϵ in favor of the summation range M of m. Besides M, only this subexpression of (3.0.1) depends on ϵ explicitly:

$$M:=\sum_{m\in\mathbb{N}}1\left\{r\left|\phi'_{\widetilde{\omega|m}}u\right|>\epsilon\right\}\le\sum_{m\in\mathbb{N}}1\left\{s_{\max}^m>\frac{\epsilon}{r}\right\}\le 1+\ln\frac{\epsilon}{r}/\ln s_{\max},$$

$$\frac{1}{M}\frac{M}{\ln\frac{\epsilon_{\max}}{\epsilon}}\le\frac{1}{M}\left(\frac{1}{\ln 2}+\frac{1}{|\ln s_{\max}|}\frac{\ln r-\ln\epsilon}{\ln\epsilon_{\max}-\ln\epsilon}\right)\le\frac{1}{M}(c_1+c_2|\ln r|).$$

Secondly, we can estimate $\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}$'s integrand $1_{(R(x,1),R(x,0)]}\left(\frac{r}{|\psi' x|}\right)\frac{r^{D-k}A(|\psi x-\psi z|,r)}{\int|\psi' y|^D A(|\psi y-\psi z|,r)\,d\mu(y)}$. Lemma 4.13 item (3) "cancels" r^D with the denominator. Item (2) replaces A in the numerator with the interior of its supporting ball $B(x,ar)$:

$$\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}(x,r,1)\le 1_{(R(x,1),R(x,0)]}\left(\frac{r}{\left|\psi'_{\widetilde{\omega|m},u}x\right|}\right)C_A r^{-k}C_k^{\mathrm{var}}\left(\left(\psi_{\widetilde{\omega|m},u}F\right)_r,\psi_{\widetilde{\omega|m},u}\mathring{B}(x,ar)\right).$$

The estimate $K^{-1}\le\left|\psi'_{\widetilde{\omega|m},u}x\right|\le K$ in the indicator function will help us find a finite measure τ. We apply the above preparations to (3.0.1) and obtain the dominating function:

$$\sup_{0<\epsilon\le\epsilon_{\max}/2}\left[\frac{\rho^{k,\mathrm{var}}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(x,r,1)}{\ln\frac{\epsilon_{\max}}{\epsilon}}+\sum_{m\in\mathbb{N}}\frac{M}{\ln\frac{\epsilon_{\max}}{\epsilon}}\frac{1\left\{r\left|\phi'_{\widetilde{\omega|m}}u\right|>\epsilon\right\}}{M}\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}(x,r,1)\right]$$

$$\le\left[\sup_{\epsilon}\frac{\rho^{k,\mathrm{var}}_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(x,r,1)}{\ln\frac{\epsilon_{\max}}{\epsilon}}\right]+\left[\sup_M\frac{1}{M}\sum_{m<M}(c_1+c_2|\ln r|)\pi^{k,\mathrm{var}}_{\widetilde{\omega|m}}(x,r,1)\right]$$

$$\leq \left[\sup_{\epsilon} \frac{1_{(\max\{R(x,0),\epsilon\},\epsilon_{\max}]}(r)\, C_k^{\mathrm{var}}\left(F_r, \overset{\circ}{B}(x,ar)\right)}{\ln\frac{\epsilon_{\max}}{\epsilon}\, r^k} \right] + \left[C_A\left(c_1 + c_2 |\ln r|\right) h(r,\omega,x)\right].$$

Thirdly, the Young inequality (known from the duality of Orlicz spaces) will be used to show the h-summand in the last line is integrable w.r.t $r^{-1} dr\, d\nu_{\mathrm{bi}}(\omega,x)$. (Recall $R(x,1)/K < r \leq R(x,0)K$.) The function $C_A(c_1 + c_2|\ln r|)$ alone is “very” integrable: By (4.4.1), there is a $c_3 > 1$ such that

$$\int_F \int_{R(x,1)/K}^{R(x,0)K} c_3^{|\ln r|} \frac{dr}{r}\, d\nu(x) \leq \frac{c_3^{-\ln K}}{\ln c_3} \int_F c_3^{-\ln R(x,0)} d\nu(x) < \infty.$$

This also shows the measure $d\tau(r,\omega,x) := r^{-1} 1_{(R(x,1)/K, R(x,0)K]}(r)\, dr\, d\nu_{\mathrm{bi}}(\omega,x)$ is finite. We have concatenated $r \mapsto |\ln r|$ with $t \mapsto \Psi(t) := \min\left\{t^2, c_3^t\right\}$, and $\Psi(|\ln r|)$ stays integrable. The function Ψ is convex. A straightforward computation shows its Legendre transform $\Phi(t) := \int_0^t \inf\{u : \Psi'(u) > s\}\, ds$ satisfies $\lim_{t\to\infty}(t \ln t)/\Phi(t) = \ln c_3$. So $\Phi(|h|)$ is τ-integrable due to our assumption on h. The Young inequality for Ψ and Φ (see [RR91]) yields

$$\int |\ln r|\, h(r,\omega,x)\, d\tau(r,\omega,x) \leq \int \Phi(|h|)\, d\tau + \int \Psi(|\ln r|)\, d\tau < \infty.$$

Due to linearity, $C_A(c_1 + c_2|\ln r|)h(r,\omega,x)$ is τ-integrable. So the dominating function is integrable. □

Lemma 4.18. *The uniform integrability assumption* (3.0.1) *of the theorem holds if:*

(1)
$$c_{\sup} := \operatorname{ess\,sup}_{\hat r > 0, \hat x \in F} \hat r^{-k} C_k^{\mathrm{var}}\left(F_{\hat r}, \overset{\circ}{B}(\hat x, a\hat r)\right) < \infty$$
(μ-essential in $\hat x$, Lebesgue in $\hat r$) or

(2) *$k = d$ (Minkowski content) or*

(3) *$k = d - 1$ (surface content).*

Proof. We will reduce the first assertion to the preceding Lemma 4.17. Define $\hat x := \phi_{\widetilde{\omega|n}} x$, $\hat r := r\left|\phi'_{\widetilde{\omega|n}} u\right|$. The proof of Lemma 4.14 (localizing and scaling $\pi^{k,\mathrm{var}}_{\widetilde{\omega|n}} = \rho^{k,\pm}_{\dots}$) shows

$$r^{-k} C_k^{\mathrm{var}}\left(\left(\psi_{\widetilde{\omega|m,u}} F\right)_r, \psi_{\widetilde{\omega|m,u}} \overset{\circ}{B}(x,ar)\right) = \hat r^{-k} C_k^{\mathrm{var}}\left(F_{\hat r}, \overset{\circ}{B}(\hat x, a r)\right) \leq c_{\sup}$$

whenever $r \leq \left|\psi'_{\widetilde{\omega|m,u}} x\right| R(x,0)$. This and then $K^{-1} \leq \left|\psi'_{\widetilde{\omega|m,u}} x\right| \leq K$ imply for the preceding Lemma's h (4.7.1),

$$h(r,\omega,x) \leq \sup_{M>0} \frac{1}{M} \sum_{m=0}^{M-1} c_{\sup}\, 1_{(R(x,1),R(x,0)]}\left(\frac{r}{\left|\psi'_{\widetilde{\omega|m,u}} x\right|}\right) \leq c_{\sup}\, 1_{(R(x,1)/K, R(x,0)K]}(r).$$

The measure $d\tau(r,\omega,x) := r^{-1} 1_{(R(x,1)/K,R(x,0)K]}(r)\, dr\, d\nu_{\mathrm{bi}}(\omega,x)$ is finite, see Lemma 4.11 (5). As a constant function, $c_{\sup}$ belongs to its Zygmund space.

The other assertions reduce to the first. In case $k = d$ (volume), the supremum condition is trivial. The case $k = d - 1$ (surface area) is proved in [RZ10, Remark 3.2.2]. (Although stated only for self-similar F, the proof works for general compact sets.) □

5. APPENDIX: VECTOR TOEPLITZ THEOREM

Lemma 5.1. *Let α_n, α_∞ be finite, signed Borel measures on $\mathbb{R}^d$, $n \in \mathbb{N}$, such that the weak*-limit (point-wise limit on continuous functions)*

$$\operatorname{wlim}_{n\to\infty} \alpha_n = \alpha_\infty$$

exists. Let b_n, b_∞ be a bounded family of equicontinuous functions whose averages converge in supremum norm

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} b_n = b_\infty.$$

Then the following limit exists:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \alpha_n(b_n) = \alpha_\infty(b_\infty).$$

Proof. We will prove[2] the assertion in only one dimension for simplicity and because product functions are dense. All measures shall be supported in $[x, y]$, neglecting tails as the family is tight. Due to the Banach-Steinhaus theorem, $\sup_n \|\alpha_n\|_{C'[x,y]} < \infty$. Since the quality of approximation is determined by the norm and the modulus of continuity only ([Wer00, Thm. 1.2.10]), we can equiuniformly approximate the b_n with Bernstein polynomials p_t using a fixed, finite set T of points,

$$\left\| b_n - \sum_{t\in T} p_t b_n(t) \right\|_{C[x,y]} < \epsilon.$$

By combining both estimates, we will be finished once we can show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \alpha_n \left(\sum_{t\in T} p_t b_n(t) \right) = \alpha_\infty \left(\sum_{t\in T} p_t b_\infty(t) \right).$$

This reduces the assertion to the sequences of reals: consider $a_n := \alpha_n(p_t)$, $b_n(t)$ for each point $t \in T$ separately. There, it is a special case of Toeplitz's theorem [Wer00, Thm. 4.4.6] for the resummation method $1_{\{n\le N\}} b_n/N$ applied to a_n. ☐

6. APPENDIX: EXAMPLE

The group of Möbius transformations of $\mathbb{R}^d$ is generated by similarities and the Möbius inversion $x \mapsto x/|x|^2$. The Möbius transformation ϕ specified by its pole $p \in \mathbb{R}^d \cup \{\infty\}$ and the derivative rA ($r > 0$, $A \in O(d)$) at its fixed point $f \in \mathbb{R}^d$ is given by

$$\begin{aligned} \phi x &= f + rA(x-f) + O\left(\left|x-f\right|^2\right) \\ &= f + rA\left(\mathrm{id} - 2\,\mathrm{Proj}_{\frac{p-f}{|p-f|}}\right)\left(\frac{\left|p-f\right|^2}{\left|p-x\right|^2}(x-p) + (p-f)\right), \end{aligned} \tag{6.0.1}$$

where Proj_y denotes the orthogonal projection onto y.

The example in the introduction is generated by three Möbius maps. They agree with those of the classical Sierpinski triangle up to first order at their fixed points, except we reduce the contraction ratio to re-obtain the Open Set Condition. All three maps ϕ_i share $r_i = 785/2000$, $A_i = \mathrm{id}$ for $i \in I = \{1, 2, 3\}$. But $f_1 = (0, 0)$, $p_1 = \left(3, \frac{5}{2}\right)$ and $f_2 = (1, 0)$, $p_2 = (-3, -2)$ and $f_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $p_3 = (-2, -4)$ differ.

[2]A direct proof is easier than checking the topological assumptions of [CP69].

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